Statistics of Three Dimensional Random Voronoi Tessellations

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1 Introduction

If S is a set of points in a Euclidean space R^n, and each point of the space is associated with the nearest point of S, then the space is divided into convex polyhedra, or *cells*. Such a partition is called a *Voronoi* tessellation, also known as a *Dirichlet* or *Theissen tessellation*. When S is generated randomly, the result is a *random Voronoi tessellation*. Such patterns turn up in the crystallization of metals [1,2], geography [3], pattern recognition [4], numerical interpolation [5], and many other subjects.

A general scheme was derived in [6] for calculating statistics of random Voronoi tessellations for sets S generated by a Poisson point process of unit density. This scheme is applied in this paper to find statistics of random tessellations of three dimensional space and plane cross-sections of such tessellations.

Meijering [1] derived the mean values of many quantities. These are given in Table 1. Gilbert [2] expressed the variance of the cell volume in terms of a double integral. This paper finds the variances and covariances for all these quantities in terms of integrals, which are evaluated numerically. Also found are the distributions of edge lengths of cells and cross-sections.

2 Tessellation geometry

Henceforth, the members of S will be called *seeds* (due to their role in generating cells), and *point* will refer to a general point in the space, usually specified to be one of the types mentioned above. The open ball whose center is at a point and which has the nearest neighbor seeds of that point on its circumference will be called the *void* of the point.

The points closer to one seed than another are divided by the perpendicular bisecting plane between the two seeds. Thus a cell is the intersection of half-spaces, and hence is convex. For a random tessellation, only a finite number of half-planes bound the cell, so the cell is a convex polyhedron. It follows that a plane cross-section of a 3-dimensional tessellation is a tessellation of the plane composed of convex polygons.

The points of a tessellation of 3-space plane are of four types, depending on how many nearest neighbors in S they have. A point with exactly one nearest neighbor is in the interior of a cell, a point with two nearest neighbors is on the face between two cells, a point with three nearest neighbors is on an edge shared by three cells, and a point with four neighbors is a vertex where three cells meet. There is zero probability that there will be any point with five or more nearest neighbors.

Likewise, the points of a plane cross-section are of three types, depending on how many nearest neighbors in S they have. A point with exactly one nearest neighbor is in the interior of a cell, a point with two nearest neighbors is on the boundary between cells, and a point with three nearest neighbors is a vertex where three cells meet.

The central object of this paper is a configuration of seeds and points. The definition of a configuration type consists of

1. the number m of seeds involved,

2. the number k of points and their types, and

3. a specification of which seeds $S_0, \ldots, S_{m-1}$ are nearest neighbors of which points $P_1, \ldots, P_k$.

Usually there will be one or two points and their nearest seeds. An example configuration would be a vertex and its four neighbor seeds. Note that an actual instance of a configuration in a tessellation may have several possible labellings. For example, a vertex configuration has 24 possible labellings of its seeds.

A *complete configuration* is a configuration that includes all the neighbor seeds of its points. Examples would be a vertex and its four neighbor seeds, or an edge point with its three neighbor seeds. An example of an incomplete configuration would be a vertex and one neighboring seed. For a complete configuration, item 3 above can be rephrased in two parts:

3a. a specification of the geometrical relation of points to seeds (i.e., on perpendicular bisector, at circumcenter), and

3b. a requirement that the voids of the points (as defined by 3a) are empty of seeds.

A set of m seeds and k (untyped) points that satisfies the geometrical relationships 3a will be called a *potential configuration*. If the condition 3b is also true, it is an *actual configuration*, and the points are necessarily of the requisite types. As an example, consider the configuration of a vertex and its four neighbor seeds. A potential configuration would be any four seeds and their circumcenter. This is an actual configuration if the void of the configuration is empty of other seeds.

The method of this paper may be outlined as follows:

1. State the problem in terms of an incomplete configuration.

2. Embed the incomplete configuration in a complete configuration.

3. In the space of all potential configurations, write down the expectation measure for all potential configurations defined by the Poisson process generating the seeds.

4. Multiply by the probability that the void region is empty to get the expectation measure for actual configurations.

5. Integrate over some variables to find the induced measure on the space of incomplete configurations.

6. Solve the original problem.

3 Configuration spaces and measures

Introduce a canonical parameter space W for all complete configurations of a given type as follows. Let the seeds have locations $S_0, \ldots, S_{m-1}$. The seed coordinates thus form a space $W_S = (\mathrm{R}^n)^m$. A point P_i of the k-dimensional skeleton of the tessellation is on the k-plane through the circumcenter of its neighbor seeds and perpendicular to the $(n-k)$-plane those seeds determine. Let y_i represent the coordinates of P_i in this plane. These point parameters y_i form a space $W_P = \mathrm{R}^q$ for some q, which may be zero if all the points are vertices. Then $W = W_S \times W_P$.

Λ tessellation T generates in W a set W_T consisting of all the instances of the configuration occurring in T. Different labellings are considered different instances. Let $d\mu_T$ be Hausdorff measure of dimension q restricted to W_T. The configuration measure $d\mu$ will be the expectation of $d\mu_T$ under the probability measure on the space of tessellations T defined by the Poisson process generating S.

For potential configurations, $d\mu_T$ is the product of a sum of unit point measures on W_S (one for each ordered subset of m seeds of S) and Lebesgue measure on W_Γ. By the unit density of the Poisson process, the expectation of the sum of the unit point measures is Lebesgue measure on W_S. Hence the potential configuration measure $d\mu_{pot}$ is Lebesgue measure on W,

$$d\mu_{pot} = dS_0 \ldots dS_{m-1} dy_1 \ldots dy_k. \tag{3.1}$$

The actual configuration measure $d\mu$ will be nonzero only on the subdomain W_0 of W for which none of the seeds of the configuration are in any of the voids. In W_0, the probability that the voids will be empty of other seeds of S is the Poisson factor e^{-A}, where A is the n-dimensional measure of the union of the voids. Hence

$$d\mu = e^{-A} dS_0 \ldots dS_{m-1} dy_1 \ldots dy_k \qquad \text{restricted to } W_0. \tag{3.2}$$

It will usually be convenient to change to coordinates relative to S_0 for $S_1, \ldots, S_{m-1}$. Note the Jacobian of this transformation is 1. Also, results will often be in terms of expected value per cell. For this, we may assume $S_0 = 0$ and factor S_0 out of W, leaving parameter space W', and factor off the dS_0 part of $d\mu$, which leaves the expected measure $d\sigma$ for configurations associated with a single cell (that generated by S_0):

$$d\sigma = e^{-A} dS_1 \ldots dS_{m-1} dy_1 \ldots dy_k \qquad \text{restricted to } W_0'. \tag{3.3}$$

The applicability of this scheme to calculate a statistic or distribution depends on whether it can be phrased in terms of configurations of a small number of points. Two points is the maximum in this paper. Hence the edge length distribution is calculable, but the face area distribution is not.

4 Second order statistics

Second order statistics, such as the expected square of cell volume, can be calculated by finding the measure of pairs of points (P_1, P_2) associated with the cell generated by S_0. Let σ_c be the configuration measure for the common seeds of P_1 and P_2, and let σ_i be the measure for the remaining seeds of P_i. Then the configuration measure may be written

$$d\sigma = e^{-A} d\sigma_c d\sigma_1 d\sigma_2, \tag{4.1}$$

where A is the void area.

Pairs may be classified by the types of points and by how many of their defining seeds the points have in common. Pairs with S_0 in common belong to the same cell, pairs with two seeds in common belong to the same face, and pairs with three seeds in common belong to the same edge. For computational purposes, it will be most convenient to classify pairs primarily by the number of common seeds and secondarily by the point types. This way, for each σ_c, a set of σ_i's may be constructed and considered in all possible pairs.

5 Three common seeds and edge length distribution

The only interesting pair with three common seeds is a pair of vertices at the ends of an edge. By integrating over all variables in the configuration measure except the edge length, we will get the distribution function of edge lengths. Let S_0, S_1, and S_2 be the common seeds, and let S_3 and S_4 be the other determining seeds of P_1 and P_2 respectively. The common measure is

$$d\sigma_c = dS_1 dS_2. \tag{5.1}$$

Let S_1 have spherical coordinates (r_1, ϕ_1, θ_1), and let S_2 have spherical coordinates (r_2, ϕ_2, θ_2) in a system whose north polar axis contains S_1. Then

$$d\sigma_c = r_1^2 \sin\phi_1 dr_1 d\phi_1 d\theta_1 r_2^2 \sin\phi_2 dr_2 d\phi_2 d\theta_2. \tag{5.2}$$

Let Q be the circumcenter of S_0, S_1, and S_2. P_1 and P_2 then lie on the line through Q perpendicular to the circumcircle. Let S_3 have spherical coordinates (r_3, ϕ_3, θ_3) in a system whose north polar axis contains Q and has P_1 at $\theta_3 = 0$. Let S_4 have spherical coordinates (r_4, ϕ_4, θ_4) in a system whose north polar axis contains Q and has P_2 at $\theta_4 = 0$. Then

$$\begin{aligned} d\sigma_1 &= r_3^2 \sin\phi_3 dr_3 d\phi_3 d\theta_3, & (5.3) \\ d\sigma_2 &= r_4^2 \sin\phi_4 dr_4 d\phi_4 d\theta_4. & (5.4) \end{aligned}$$

Note that each edge is counted twice, once for each orientation.

Some changes of variables will permit integrals to be done. First, replace (r_1, r_2, ϕ_2) with (z, β_1, β_2) where z is the distance from S_0 to Q, β_1 is the angle QS_0S_1, and β_2 is the angle QS_0S_2, oriented so that $\phi_2 = \beta_2 - \beta_1$. Then

$$d\sigma_c = 64z^5 \cos^2\beta_1 \cos^2\beta_2 \sin^2(\beta_2 - \beta_1) \sin\phi_1 dz d\beta_1 d\beta_2 d\phi_1 d\theta_1 d\theta_2. \tag{5.5}$$

The domains are

$$0 \le z < \infty, \qquad -\pi/2 < \beta_1 < \beta_2 < \pi/2, \qquad 0 \le \theta_1, \qquad \theta_2 < \pi, \qquad 0 \le \phi_1 \le \pi/2. \tag{5.6}$$

It is possible to now integrate over β_1, β_2, θ_1, θ_2, and ϕ, giving an effective common measure

$$d\sigma_c = 12\pi^4 z^6 dz. \tag{5.7}$$

This includes the correction for double counting.

In $d\sigma_1$, we replace r_3 by β_1, which is the angle QS_0P_1. Then

$$d\sigma_1 = 8z^3 \sin^2\phi_3 \cos\theta_3 \sec^2\beta_1 (\cos\phi_3 + \sin\phi_3 \tan\beta_1 \cos\theta_3)^2 d\beta_1 d\theta_3 d\phi_3, \tag{5.8}$$

with domain given by

$$-\pi/2 < \beta_1 < \pi/2, \quad -\pi/2 < \theta_3 < \pi/2, \quad -\tan\beta_1 \cos\theta_3 \le \cot\phi_3 < \infty. \tag{5.9}$$

The same can be done for σ_2, except that the domain for β_2 must be

$$-\beta_1 < \beta_2 < \pi/2 \tag{5.10}$$

in order to keep the seeds out of the void interior.

The volume A of the void is

$$A = \pi z^3 B \tag{5.11}$$

where the quantity B (which will often occur in the rest of this paper) is

$$B = \sec^3\beta_1\left(\frac{2}{3}+\sin\beta_1-\frac{\sin^3\beta_1}{3}\right)+\sec^3\beta_2\left(\frac{2}{3}+\sin\beta_2-\frac{\sin^3\beta_2}{3}\right). \tag{5.12}$$

All variables except β_1, β_2, and z may be integrated analytically to give a reduced configuration measure

$$d\sigma = 12\pi^6 z^{11} e^{-\pi z B}\sec^2\beta_1(\sec\beta_1+\tan\beta_1)^2\sec^2\beta_2(\sec\beta_2+\tan\beta_2)^2 d\beta_1 d\beta_2 dz. \tag{5.13}$$

The distribution of edge lengths L may be found by replacing z by $z = L/(\tan\beta_1+\tan\beta_2)$ and integrating over β_1 and β_2:

$$\begin{aligned} f_L(L) \quad = \quad & 12\pi^6 L^{11}\int_{-\pi/2}^{\pi/2}\int_{-\beta_1}^{\pi/2} e^{-\pi L^3 B(\tan\beta_1+\tan\beta_2)^{-3}} \\ & \cdot(\tan\beta_1+\tan\beta_2)^{-12}\sec^2\beta_1(\sec\beta_1+\tan\beta_1)^2\sec^2\beta_2(\sec\beta_2+\tan\beta_2)^2 d\beta_1 d\beta_2. \end{aligned} \tag{5.14}$$

An alternate, less singular form is

$$\hat{B} = \cos^3\beta_2\left(\frac{2}{3}+\sin\beta_1-\frac{\sin^3\beta_1}{3}\right)+\cos^3\beta_1\left(\frac{2}{3}+\sin\beta_2-\frac{\sin^3\beta_2}{3}\right), \tag{5.15}$$

$$\begin{aligned} f_L(L) \quad = \quad & 12\pi^6 L^{11}\int_{-\pi/2}^{\pi/2}\int_{-\beta_1}^{\pi/2} e^{-\pi L^3\hat{B}/\sin^3(\beta_1+\beta_2)} \\ & \cdot\sin^{-12}(\beta_1+\beta_2)\cos^8\beta_1(1+\sin\beta_1)^2\cos^8\beta_2(1+\sin\beta_2)^2 d\beta_1 d\beta_2. \end{aligned} \tag{5.16}$$

This may be normalized to a probability density function by dividing by the total number of edges, $144\pi^2/35$. The normalized distribution is tabulated in table 1 and graphed in figure 1.

The moments of the distribution may be expressed as double integrals by integrating over L:

$$\begin{aligned} E(L^n) \quad = \quad & \frac{35}{36}\pi^{-n/3}\Gamma\left(4+\frac{n}{3}\right)\int_{-\pi/2}^{\pi/2}\int_{-\beta_1}^{\pi/2} B^{-4-n/3} \\ & \cdot(\tan\beta_1+\tan\beta_2)^n\sec^2\beta_1(\sec\beta_1+\tan\beta_1)^2\sec^2\beta_2(\sec\beta_2+\tan\beta_2)^2 d\beta_1 d\beta_2. \end{aligned} \tag{5.17}$$

or

$$\begin{aligned} E(L^n) \quad = \quad & \frac{35}{36}\pi^{-n/3}\Gamma\left(4+\frac{n}{3}\right)\int_{-\pi/2}^{\pi/2}\int_{-\beta_1}^{\pi/2} \hat{B}^{-4-n/3} \\ & \cdot\sin^n(\beta_1+\beta_2)\cos^8\beta_1(1+\sin\beta_1)^2\cos^8\beta_2(1+\sin\beta_2)^2 d\beta_1 d\beta_2. \end{aligned} \tag{5.18}$$

Values of interest are:

$$E(L) = \frac{7}{9}\left(\frac{3}{4\pi}\right)^{1/3}\Gamma(4/3) = 0.430857994283959 \tag{5.19}$$

$$E(L^2) = 0.290877746899549 \tag{5.20}$$

$$\mathrm{Var}(L) = 0.105239135661153 \tag{5.21}$$

with the exact value for $E(L)$ following from Meijering's results.

6 Two common seeds

These pairs are on the same face, but not on the same edge. Let S_0 and S_1 be the common seeds. The common measure is

$$d\sigma_c = dS_1. \tag{6.1}$$

Let S_1 have spherical coordinates (r_1, ϕ_1, θ_1). Then

$$d\sigma_c = r_1^2 \sin\phi_1 dr_1 d\phi_1 d\theta_1. \tag{6.2}$$

Face point. The remaining measure for a face area point P_i is Lebesgue measure on the perpendicular bisector plane of S_0S_1. Let $(\rho,\ \theta)$ be the polar coordinates of P_i on this plane. Then

$$d\sigma_i = \rho d\rho d\theta. \tag{6.3}$$

To replace ρ and θ, introduce coordinates

$$\begin{aligned} \omega &= \text{angle } S_1S_0P_i \\ \theta_P &= \text{longitude of } P_i. \end{aligned} \tag{6.4}$$

Then

$$d\sigma_i = \frac{1}{4} r_1^2 \tan\omega \sec^2\omega d\theta_P d\omega. \tag{6.5}$$

Edge point. The remaining measure for an edge point P_i is the product of Lebesgue 3-measure for S_2 and linear measure for P_i along the centerline of S_0, S_1, and S_2:

$$d\sigma_i = dS_2 dy \tag{6.6}$$

where y is the distance of P_i along the centerline from the circumcenter. Let (r_2, ϕ_2, θ_2) be the spherical coordinates of S_2 with S_1 northerly. Then

$$d\sigma_i = r_2^2 \sin\phi_2 dr_2 d\phi_2 d\theta_2 dy. \tag{6.7}$$

To replace r_2, θ_2, and y, introduce coordinates

$$\begin{aligned} \omega &= \text{angle } S_1S_0P_i \\ \theta_P &= \text{longitude of } P_i \\ \theta &= \text{longitude from } S_2 \text{ to } P_i \end{aligned} \tag{6.8}$$

Then

$$d\sigma_i = \frac{1}{2} r_1^4 (\tan\omega\cos\theta\sin\phi_2 + \cos\phi_2)^2 \sin\phi_2 \sec^2\omega \tan\omega d\omega d\theta d\theta_P d\phi_2. \tag{6.9}$$

Suppose the void sphere intersection defines the domain of θ to be $-\alpha < \theta < \pi - \alpha$. The domain of ϕ_2 is $0 < \phi_2 < \operatorname{arccot}(-\tan\omega\cos\theta)$. Then we can integrate over ϕ_2 and θ to get a reduced measure

$$d\sigma_i = \frac{1}{16} r_1^4 F_E(\alpha, \omega) \sec^2\omega \tan\omega d\omega d\theta_P. \tag{6.10}$$

where

$$F_E(\alpha, \omega) = \int_{-\alpha}^{\pi-\alpha} (3\tan^2\omega\cos^2\theta + 1)\operatorname{arccot}(-\tan\omega\cos\theta) + 3\tan\omega\cos\theta d\theta. \tag{6.11}$$

Vertex point. The remaining measure for a vertex P_i is the product of Lebesgue 3-measure for S_2 and S_3:

$$d\sigma_i = dS_2 dS_3. \tag{6.12}$$

Let (r_2, ϕ_2, θ_2) and $(r_2,\ \phi_2,\ \theta_2)$ be the spherical coordinates of S_2 and S_3 with S_1 northerly. Then

$$d\sigma = r_2^2 \sin\phi_2 dr_2 d\phi_2 d\theta_2 r_2^2 \sin\phi_2 dr_2 d\phi_2 d\theta_2. \tag{6.13}$$

To replace r_2, θ_2, r_3, and θ_3, introduce coordinates

$$\begin{aligned} \omega &= \text{angle } S_1 S_0 P: \\ \theta_P &= \text{longitude } of\ P_i \\ \theta_A &= \text{longitude from } P_i \text{ to } S_2 \\ \theta_B &= \text{longitude from } P_i \text{ to } S_3. \end{aligned} \tag{6.14}$$

Then

$$\begin{aligned} d\sigma &= r_1^6(\tan\omega\cos\theta_A\sin\phi_2+\cos\phi_2)^2\sin^2\phi_2(\tan\omega\cos\theta_B\sin\phi_3+\cos\phi_3)^2 \\ &\quad \cdot\sin^2\phi_3\sin(\theta_B-\theta_A)\sec^2\omega\tan\omega d\omega d\theta d\theta_P d\phi_2. \end{aligned} \tag{6.15}$$

The domain for immediate integration is

$$0<\phi_2<\operatorname{arccot}(-\tan\omega\cos\theta_A),\quad 0<\phi_3<\operatorname{arccot}(-\tan\omega\cos\theta_B),\quad -\alpha<\theta_A<\theta_b<\pi-\alpha. \tag{6.16}$$

Then we can now integrate over ϕ_2, ϕ_3, θ_A, and θ_B to get a reduced measure

$$d\sigma = \frac{1}{64} r_1^6 F_V(\alpha,\omega)\sec^2\omega\tan\omega d\omega d\theta_P. \tag{6.17}$$

where

$$F_V(\alpha,\omega) = \int_{-\alpha}^{\pi-\alpha} f(s)s(F(-\cos\alpha)-2F(s)+F(\cos\alpha))d\alpha, \tag{6.18}$$

$$f(s) = (3\tau^2 s^2+1)\operatorname{arccot}(-\tau s)+3\tau s, \tag{6.19}$$

$$F(s) = (\tau^2 s^2+s)\operatorname{arccot}(-\tau s)+\tau s62, \tag{6.20}$$

$$s = \cos\theta, \tag{6.21}$$

$$\tau = \tan\omega \tag{6.22}$$

General form. The general form of the total configuration measure is

$$d\sigma = e^{-A}c_1 r_1^{n_1}F_1(\alpha_1,\omega_1)\sec^2\omega_1\tan\omega_1 d\omega_1 d\theta_{P1} c_2 r_1^{n_2} F_2(\alpha_2,\omega_2)\sec^2\omega_2\tan\omega_2 d\omega_2 d\theta_{P2} r_1^2\sin\phi_1 dr_1 d\phi_1 d\theta_1, \tag{6.23}$$

where c_i and n_i are constants depending on the point type, and F_i is a function depending on the point type, as displayed in the formulas above. Replace $(\theta_1,\theta_2,\omega_1,\omega_2)$ by $(\theta_Q,\alpha_1,\alpha_2,\gamma)$ where

$$\theta_{P1} = \theta_Q-\alpha_1, \tag{6.24}$$

$$\theta_{P2} = \theta_Q+\alpha_2, \tag{6.25}$$

$$\tan\omega_1 = \tan\gamma\sec\alpha_1, \tag{6.26}$$

$$\tan\omega_2 = \tan\gamma\sec\alpha_2 \tag{6.27}$$

Note that α_1 and α_2 are the same quantities introduced previously. Then

$$
\begin{aligned}
d\sigma &= e^{-A} c_1 c_2 r_1^{n_1+n_2+2} F_1(\alpha_1,\gamma) F_2(\alpha_2,\gamma) \sec^3\alpha_1 \sec^3\alpha_2 \sin(\alpha_1+\alpha_2) \\
&\quad \cdot \tan^3\gamma \sec^2\gamma d\alpha_1 d\alpha_2 d\gamma d\theta_Q \sin\phi_1 dr_1 d\phi_1 d\theta_1. \qquad (6.28)
\end{aligned}
$$

An abuse of notation has been committed with the F's, letting the argument indicate a formally different function. Integrating over θ_Q, ϕ_1, and θ_1 gives a factor of $8\pi^2$:

$$
d\sigma = 8\pi^2 e^{-A} c_1 c_2 r_1^{n_1+n_2+2} F_1(\alpha_1,\gamma) F_2(\alpha_2,\gamma) \sec^3\alpha_1 \sec^3\alpha_2 \sin(\alpha_1+\alpha_2) \tan^3\gamma \sec^2\gamma d\alpha_1 d\alpha_2 d\gamma. \qquad (6.29)
$$

We make a further change to bring the set of variables into conformity with the standard set used in this paper. Replace (r_1,α_1,α_2) with $(z,\ \beta_1,\beta_2)$, where

$$
\begin{aligned}
r_1 &= 2z\cos\gamma, && (6.30)\\
\tan\alpha_1 &= \tan\beta_1/\sin\gamma, && (6.31)\\
\tan\alpha_2 &= \tan\beta_2/\sin\gamma. && (6.32)
\end{aligned}
$$

Then

$$
d\sigma = 2^{r\iota_1+n_2+6}\pi^2 c_1 c_2 e^{-\pi B z^3} F_1(\beta_1,\gamma) F_2(\beta_2,\gamma)(\tan\beta_1+\tan\beta_2)\cos^{n_1+n_2-2}\gamma \sec^2\beta_1 \sec^2\beta_2 d\beta_1 d\beta_2 d\gamma dz. \qquad (6.33)
$$

Again, the factor functions have formally changed. The quantity B is the same as before.

The z integral can be done analytically, leaving

$$
\begin{aligned}
d\sigma &= \frac{1}{3} 2^{n_1+n_2+6}\pi^{1+(n_1+n_2)/3}\Gamma((n_1+n_2)/3+1) c_1 c_2 B^{-(n_1+n_2)/3-1} \qquad (6.34)\\
&\quad \cdot F_1(\beta_1,\gamma) F_2(\beta_2,\gamma)(\tan\beta_1+\tan\beta_2)\cos^{n_1+n_2-2}\gamma \sec^2\beta_1 \sec^2\beta_2 d\beta_1 d\beta_2 d\gamma.
\end{aligned}
$$

The domain is

$$
0<\gamma<\pi/2, \quad -\pi/2<\beta_1<\pi/2, \quad -\beta_1<\beta_2<\pi/2. \qquad (6.35)
$$

References:

[1] J. L. Meijering, "Interface area, edge length, and number of vertices in crystal aggregates with random nucleation," *Philips Res. Rep.* **8** (1953), 270-290.

[2] E. N. Gilbert, "Random subdivisions of space into crystals." *Annals of Mathematical Statistics* **33** (1962), 958–972.

[3] D. Rhynsburger, "Analytic delineation of Theissen polygons." *Geographical Analysis* **5** (1973), 133–141.

[4] R. Sibson, "The Dirichlet tessellation as an aid in data analysis." *Scandinavian J. Statistics* **7** (1980), 14–20.

[5] C. L. Lawson, "Software for C^1 surface interpolation." In *Mathematical Software, III.* Academic Press, New York, 1977, pp. 161–193.

[6] K. A. Brakke, "Statistics of Random Plane Voronoi Tessellations." (unpublished)

Table 1. Probability density function f_L of edge lengths L of a random Voronoi tessellation in three dimensions generated by a unit density Poisson process.

L	f_L	L	f_L
0.00	1.616909054181E0	2.00	2.003578075652E-4
0.05	1.550881449610E0	2.05	1.060044075479E-4
0.10	1.484071056779E0	2.10	5.431258847951E-5
0.15	1.416509590558E0	2.15	2.692677052292E-5
0.20	1.348240491066E0	2.20	1.290698672091E-5
0.25	1.279321470737E0	2.25	5.976860605026E-6
0.30	1.209827585793E0	2.30	2.671658628863E-6
0.35	1.139854904429E0	2.35	1.151864022629E-6
0.40	1.069524802906E0	2.40	4.786163440079E-7
0.45	9.989888343732E-1	2.45	1.915112401722E-7
0.50	9.284339742239E-1	2.50	7.373546817565E-8
0.55	8.580878512108E-1	2.55	2.729540818461E-8
0.60	7.882233429050E-l	2.60	9.707105559146E-9
0.65	7.191616841676E-1	2.65	3.313854010528E-9
0.70	6.512730627001E-1	2.70	1.085118767444E-9
0.75	5.849736212586E-1	2.75	3.405482144160E-10
0.80	5.207179127395E-1	2.80	1.023516387349E-10
0.85	4.589862011435E-1	2.85	2.943632114733E-11
0.90	4.002665676502E-1	2.90	8.094736184960E-12
0.95	3.450325141577E-l	2.95	2.126720351996E-12
1.00	2.937175518087E-1	3.00	5.334145383441E-13
1.05	2.466889760644E-1	3.05	1.276216867384E-13
1.10	2.042235088008E-1	3.10	2.910367334435E-14
1.15	1.664876024323E-1	3.15	6.321111692749E-15
1.20	1.335248813054E-1	3.20	1.306532395719E-15
1.25	1.052524548101E-1	3.25	2.567951265721E-l6
1.30	8.146677643717E-2	3.30	4.795701430365E-l7
1.35	6.185851675679E-2	3.35	8.503059396260E-18
1.40	4.603477601235E-2	3.40	1.430260058539E-18
1.45	3.354608461124E-2	3.45	2.280502665965E-19
1.50	2.391517483031E-2	3.50	3.444141831032E-20
1.55	1.666451714206E-2	3.55	4.922950505153E-21
1.60	1.134006673124E-2	3.60	6.654617053004E-22
1.65	7.529443836481E-3	3.65	8.500275640637E-23
1.70	4.873707551006E-3	3.70	1.025213923257E-23
1.75	3.072797955924E-3	3.75	1.166612839121E-24
1.80	1.885465959787E-3	3.80	1.251494700702E-25
1.85	1.124990535497E-3	3.85	1.264679663993E-26
1.90	6.521728053798E-4	3.90	1.202930597858E-27
1.95	3.670293868739E-4	3.95	1.076139052200E-28

Table 2. Plane statistics:

E(area)	=	1.000000000000000
E(vertices)	=	6.000000000000000
E(edge length)	=	0.666666666666666
E(perimeter)	=	4.000000000000000
Var(area)	=	0.28017604092670
Var(edge length)	=	0.18562733470512
Var(perimeter)	=	0.94549301073850
Var(vertices)	=	1.78081169901223
Cov(area, edge)	=	0.03720310720856
Cov(perimeter,edge)	=	0.085295154692359
Cov(number, edge)	=	-0.089437445910828
Cov(area, perimeter)	=	0.49047211300708
Cov(area, vertices)	=	0.4008802046335
Cov(perimeter, vertices)	=	0.65058312387652

Table 3. 3D cell face statistics:

E(area)	=	0.374683050589892
E(vertices)	=	5.227573437889669
E(edge length)	=	0.430857994283959
E(perimeter)	=	2.252341806421243
Var(area)	=	0.14238966950293
Var(edge length)	=	0.10523913566115
Var(perimeter)	=	1.4699757822866
Var(vertices)	=	2.48464067596800
Cov(area, edge)	=	0.044445918551
Cov(perimeter, edge)	=	0.16370992196
Cov(number, edge)	=	0.06789595512
Cov(area, perimeter)	=	0.42458077312504
Cov(area, vertices)	=	0.44617129643
Cov(perimeter, vertices)	=	1.42545838971

Table 4. 3D plane cross section statistics:

E(area)	=	0.685887514859605
E(vertices)	=	6.000000000000000
E(edge length)	=	0.522612294369352
E(perimeter)	=	3.135673766216111
Var(area)	=	0.22695289959512
Var (edge length)	=	0.13040733669562
Var(perimeter)	=	1.4773161590933
Var(vertices)	=	2.8271549519400
Cov(area, edge)	=	0.0379654533346
Cov(perimeter, edge)	=	0.1159756193049
Cov(number, edge)	=	-0.000131629712
Cov(area, perimeter)	=	0.54481761088193
Cov(area, vertices)	=	0.60661583030
Cov(perimeter, vertices)	=	1.49530053480

Table 5. 3D cell statistics:

perimeter = total of edge lengths;
surface = total surface area;

E(vertices)	=	27.070914928702240
E(surface)	=	5.820872595052579
E(volume)	=	1.000000000000000
E(perimeter)	=	17.495580164418480
E(edge length)	=	0.430857994283959
E (number edges)	=	40.606372393053360
E(number faces)	=	15.53545746435112
Var(volume)	=	0.179032437845
Var(surface)	=	2.191483455281431
Var(perimeter)	=	13.617940052235990
Var(vertices)	=	44.498388684999960
Cov(volume, edge)	=	0.002986565580
Cov(volume, vertices)	=	2.0777760030
Cov(volume, perimeter)	=	1.464113195990
Cov(volume, surface)	=	0.61514448699074
Cov(volume, face area)	=	0.014540335351
Cov(surface, edge)	=	0.01253075318
Cov(surface, face area)	=	0.0563219082
Cov(surface, perimeter)	=	5.050452528193
Cov(surface, vertices)	=	7.0272559379000
Cov (perimeter, edge)	=	-0.0073423626
Cov (perimeter, face area)	=	0.06543396689
Cov (perimeter, vertices)	=	21.532364020
Cov (vertex, edge)	=	-0.1779613716
Cov (vertex, face area)	=	-0.0842678804