Random Voronoi Tessellations in Arbitrary Dimension

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Abstract

Voronoi tessellations generated by Poisson point processes in n-dimensional Euclidean space are studied. Formulas for the expected measure of the k-dimensional skeleton of the tessellation are developed, along with formulas for q-dimensional cross sections. As n goes to infinity with q fixed, there is a limiting tessellation process, which is intuitively a finite dimensional cross section of an infinite dimensional tessellation.

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1 Introduction

One way to divide a space into convex polyhedra is to start with a set S of points and associate with each point P of S the set of points of the space that are closer to P than to any other point of S. The closure of this set is the *Voronoi polyhedron* of P. All the Voronoi polyhedra together form the *Voronoi tessellation* of the space generated by S. A *random Voronoi tessellation* is one generated by a random set S. In this paper, S will be assumed to be generated by a Poisson point process with unit density. The polyhedra will also be called *cells*, and the points of S will be called *seeds* due to their role in generating cells.

The k-dimensional skeleton of the tessellation is composed of points that are in the intersection of $n-k+1$ polyhedra, that is, points that have $n-k+1$ nearest neighbors in S. The expected values of the measures of the skeletons of 2 and 3 dimensional tessellations were derived by Meijering [1] and Gilbert [2]. This paper finds the expected values for arbitrary dimension, and for cross-sections of arbitrary dimension. In the case of a fixed-dimensional section, it is possible to make sense of the limit as the dimension of the parent space goes to infinity.

2 Main result

The method employed here is the same as that developed in Brakke [3]. Each type of point (interior, vertex, boundary, etc.) is characterized as being part of a configuration of seeds. The expected measure of such configurations can be expressed as an integral, which can be evaluated (analytically, in this paper) to give the expected measure of a type of point.

Each point T of the k-skeleton is the center of a ball with $n-k+1$ points of S on its boundary and none in its interior. Refer to these points as $S_0, S_1, \ldots, S_m$, where m is the codimension, $m = n-k$. The ball will be called simply the *void* of T. Let M be the m-ball determined by $S_0, \ldots, S_m$, in the m-plane determined by $S_0, \ldots, S_m$, and let Y be the center of M.

The configuration for T may be parameterized by $S_0, \ldots, S_m$, and a k-dimensional parameter $\vec{y}$ giving the location of T on the k-plane through Y and perpendicular to the m-plane containing M. From general results in [3], it follows that the expected measure of the k-skeleton of the cell generated by S_0 is

$$E(n,k) = \frac{1}{m!}\int e^{-V} dS_1 \ldots dS_m d\vec{y}, \tag{2.1}$$

where V is the n-dimensional measure of the void of T. This essentially says consider all possible configurations, multiply by the Poisson factor e^{-V} for the probability the void of T being empty of other seeds, and divide by $m!$ to correct for multiple counting of T by different labellings of $S_1, \ldots, S_m$.

Note that the quantities needed for the calculation of V are the radius r of M and the distance y of T from the center of M. Convert coordinates $(S_1, \ldots, S_m)$ to coordinates (Ω, r), where Ω is a set of dimensionless parameters. Also convert $\vec{y}$ to (Φ, y), where Φ is dimensionless. Then dimensional analysis shows that

$$E(n,k) = \frac{1}{m!}\int d\Omega \cdot \int d\Phi \cdot \int\int \exp(-V(y,r)) r^{mn-1} y^{k-1} dr dy. \tag{2.2}$$

Define the quantity $Q(n,m)$ by

$$Q(n,m) = \int d\Omega. \tag{2.3}$$

We also have

$$\int d\Phi = \alpha'(k), \tag{2.4}$$

where $\alpha'(k)$ is the $(k-1)$-measure of the boundary of a unit k-ball, and

$$V(y,r) = \alpha(n)(y^2 + r^2)^{n/2}, \tag{2.5}$$

where $\alpha(n)$ is the n-measure of an n-ball. Thus

$$E(n,k) = \frac{\alpha'(k)Q(n,m)}{m!}\int_0^\infty \int_0^\infty \exp(-\alpha(n)(r^2+y^2)^{n/2}) r^{mn-1} y^{k-1} dr dy. \tag{2.6}$$

The double integral may be done analytically to yield

$$E(n,k) = \frac{\alpha'(k)Q(n,m)}{m!}\alpha(n)^{-(mn+k)/n} \frac{\Gamma\left(\frac{mn+k}{n}\right)\Gamma\left(\frac{mn}{2}\right)\Gamma\left(\frac{k}{2}\right)}{2n\Gamma\left(\frac{mn+k}{2}\right)}. \tag{2.7}$$

The special cases $k = 0$ and $k = n$ work out correctly if one takes $\Gamma(0) = 2$. The values of $Q(n,m)$ may be found by induction on m. It is convenient to let r_m be the radius of M, and to define

$$d\mu(n,m) = Q(n,m) r_m^{nm-1} dr_m, \tag{2.8}$$

that is, $d\mu(n,m)$ is $dS_1 \dots dS_m$ integrated over all dimensionless parameters. For $m = 1$, for S_1 at radius R from S_0, after integrating over angular variables,

$$d\mu(n,1) = \alpha'(n) R^{n-1} dR. \tag{2.9}$$

Substituting $r_1 = R/2$ gives

$$d\mu(n,1) = 2^n \alpha'(n) r_1^{n-1} dr_1. \tag{2.10}$$

so

$$Q(n,1) = 2^n \alpha'(n). \tag{2.11}$$

Assuming we have $d\mu(n,m)$, for $m+1$ we want

$$d\mu(n,m+1) = \int d\mu(n,m) dS_{m+1}, \tag{2.12}$$

where the integration is over all angular variables, and r_{m+1} alone remains. Relative to the center of M, let S_{m+1} have component $\vec{z}$ in the m-plane of M and $\vec{y}$ perpendicularly. Integrating over the angular coordinates of $\vec{z}$ and $\vec{y}$ leaves

$$\mu(n,m+1) = \int d\mu(n,m) \alpha'(m) z^{m-1} \alpha'(k) y^{k-1} dy, \tag{2.13}$$

where z and y are the magnitudes of $\vec{z}$ and $\vec{y}$. Make the change of variables

$$\begin{aligned} r_m &= r_{m+1} \cos\gamma, \\ z &= r_{m+1} \cos\beta, \\ y &= r_{m+1}(\sin\beta - \sin\gamma), \end{aligned} \tag{2.14}$$

$$-\pi/2 < \gamma < \beta < \pi/2$$

The Jacobian of the transformation is

$$\frac{\partial(r_m, R, y)}{\partial(r_{m+1}, \gamma, \beta)} = r_{m+1}^2 (\sin\beta - \sin\gamma). \tag{2.15}$$

Hence

$$d\mu(n,m+1) = Q(n,m) \int_{-\pi/2}^{\pi/2} \int_{\gamma}^{\pi/2} \cos^{nm-1}\gamma \cdot \alpha'(m) \cos^{m-1}\beta \cdot \alpha'(k)(\sin\beta - \sin\gamma)^k \times d\beta\, d\gamma\, r_{m+1}^{n(m+1)-1} dr_{m+1}. \tag{2.16}$$

Thus

$$Q(n,m+1) = Q(n,m)\alpha'(m)\alpha'(k) \int_{-\pi/2}^{\pi/2} \int_{\gamma}^{\pi/2} \cos^{nm-1}\gamma \cdot \cos^{m-1}\beta \cdot (\sin\beta - \sin\gamma)^k d\beta\, d\gamma. \tag{2.17}$$

A little exercise in mathematical induction shows that

$$\int_{-\pi/2}^{\pi/2} \int_{\gamma}^{\pi/2} \cos^a\gamma \cdot \cos^b\beta \cdot (\sin\beta - \sin\gamma)^c d\beta\, d\gamma = \frac{\sqrt{\pi}\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)\Gamma\left(\frac{c+1}{2}\right)\Gamma\left(\frac{a+b+2c+2}{2}\right)}{2\Gamma\left(\frac{a+c+2}{2}\right)\Gamma\left(\frac{b+c+2}{2}\right)\Gamma\left(\frac{a+b+c+2}{2}\right)} \tag{2.18}$$

So we have

$$
\begin{aligned}
Q(n,m+1) &= Q(n,m)\frac{m\pi^{m/2}}{\Gamma(\frac{m+2}{2})}\frac{k\pi^{k/2}}{\Gamma(\frac{k+2}{2})}\frac{\sqrt{\pi}\Gamma(\frac{nm}{2})\Gamma(\frac{m}{2})\Gamma(\frac{k+1}{2})\Gamma(\frac{nm-1+m-1+2k+2}{2})}{2\Gamma(\frac{nm-1+n-m+2}{2})\Gamma(\frac{m-1+n-m+2}{2})\Gamma(\frac{nm-1+m-1+k+2}{2})} \\
&= Q(n,m)\frac{m\pi^{m/2}}{\Gamma(\frac{m+2}{2})}\frac{(n-m)\pi^{(n-m)/2}}{\Gamma(\frac{n-m+2}{2})}\frac{\sqrt{\pi}\Gamma(\frac{nm}{2})\Gamma(\frac{m}{2})\Gamma(\frac{n-m+1}{2})\Gamma(\frac{nm+m+2n-2m}{2})}{2\Gamma(\frac{nm+n-m+1}{2})\Gamma(\frac{n+1}{2})\Gamma(\frac{nm+n}{2})} \\
&= Q(n,m)\frac{\pi^{(n+1)/2}m(n-m)\Gamma(\frac{nm}{2})\Gamma(\frac{m}{2})\Gamma(\frac{n-m+1}{2})\Gamma(\frac{nm-m+2n}{2})}{2\Gamma(\frac{m+2}{2})\Gamma(\frac{n-m+2}{2})\Gamma(\frac{nm+n-m+1}{2})\Gamma(\frac{n+1}{2})\Gamma(\frac{nm+n}{2})} \\
&= Q(n,m)\frac{\pi^{(n+1)/2}(n-m)\Gamma(\frac{nm}{2})\Gamma(\frac{n-m+1}{2})\Gamma(\frac{nm-m+2n}{2})}{\Gamma(\frac{n-m+2}{2})\Gamma(\frac{nm+n-m+1}{2})\Gamma(\frac{n+1}{2})\Gamma(\frac{nm+n}{2})} \qquad (2.19) \\
&= Q(n,m)\frac{\pi^{(n+1)/2}(n-m)}{\Gamma(\frac{n+1}{2})}\frac{\Gamma(\frac{nm}{2})}{\Gamma(\frac{n(m+1)}{2})}\frac{\Gamma(\frac{n-(m+1)+2}{2})}{\Gamma(\frac{n-m+2}{2})}\frac{\Gamma(\frac{n(m+1)+n-(m+1)+1}{2})}{\Gamma(\frac{nm+n-m+1}{2})} \\
&= Q(n,1)\frac{\pi^{m(n+1)/2}(n-m)\dots(n-1)}{\Gamma(\frac{n+1}{2})^m}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n(m+1)}{2})}\frac{\Gamma(\frac{n-(m+1)+2}{2})}{\Gamma(\frac{n+1}{2})}\frac{\Gamma(\frac{n(m+1)+n-(m+1)+1}{2})}{\Gamma(\frac{2n}{2})} \\
&= \frac{2^n n\pi^{n/2}}{\Gamma(\frac{n+2}{2})}\frac{\pi^{m(n+1)/2}(n-m)\dots(n-1)}{\Gamma(\frac{n+1}{2})^m}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n(m+1)}{2})}\frac{\Gamma(\frac{n-(m+1)+2}{2})}{\Gamma(\frac{n+1}{2})}\frac{\Gamma(\frac{n(m+1)+n-(m+1)+1}{2})}{\Gamma(\frac{2n}{2})} \\
&= \frac{2^n n\pi^{n/2}}{\Gamma(\frac{n+2}{2})}\frac{\pi^{m(n+1)/2}}{\Gamma(\frac{n+1}{2})^m}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n(m+1)}{2})}\frac{\Gamma(\frac{n-(m+1)+2}{2})}{\Gamma(\frac{n+1}{2})}\frac{\Gamma(\frac{n(m+1)+n-(m+1)+1}{2})}{\Gamma(n-m)} \\
&= \frac{2^{n+1}\pi^{n/2}}{1}\frac{\pi^{m(n+1)/2}}{\Gamma(\frac{n+1}{2})^{m+1}}\frac{1)}{\Gamma(\frac{n(m+1)}{2})}\frac{\Gamma(\frac{n-(m+1)+2}{2})}{1}\frac{\Gamma(\frac{n(m+1)+n-(m+1)+1}{2})}{\Gamma(n-m)} \\
&= \frac{2^{n+1}}{1}\frac{\pi^{((m+1)(n+1)-1)/2}}{\Gamma(\frac{n+1}{2})^{m+1}}\frac{1}{\Gamma(\frac{n(m+1)}{2})}\frac{\Gamma(\frac{n-(m+1)+2}{2})}{1}\frac{\Gamma(\frac{n(m+1)+n-(m+1)+1}{2})}{\Gamma(n-m)}
\end{aligned}
$$

So

$$Q(n,m) = \frac{2^{n+1}\pi^{(m(n+1)-1)/2}\Gamma(\frac{n-m+2}{2})\Gamma(\frac{nm+n-m+1}{2})}{\Gamma(\frac{n+1}{2})^m\Gamma(\frac{nm}{2})\Gamma(n-m+1)} \qquad (2.20)$$

Then from (2.7)

$$
\begin{aligned}
E(n,k) &= \frac{\alpha'(k)Q(n,m)}{m!}\alpha(n)^{-(mn+k)/n}\frac{\Gamma\left(\frac{mn+k}{n}\right)\Gamma\left(\frac{mn}{2}\right)\Gamma\left(\frac{k}{2}\right)}{2n\Gamma\left(\frac{mn+k}{2}\right)} \qquad (2.21)\\
&= \frac{k\pi^{k/2}}{\Gamma(\frac{k+2}{2})m!}\frac{2^{n+1}\pi^{(m(n+1)-1)/2}\Gamma(\frac{n-m+2}{2})\Gamma(\frac{nm+n-m+1}{2})}{\Gamma(\frac{n+1}{2})^m\Gamma(\frac{nm}{2})\Gamma(n-m+1)}\\
&\qquad \pi^{-(mn+k)/2}\Gamma\left(\frac{n+2}{2}\right)^{(mn+k)/n}\frac{\Gamma\left(\frac{mn+k}{n}\right)\Gamma\left(\frac{mn}{2}\right)\Gamma\left(\frac{k}{2}\right)}{2n\Gamma\left(\frac{mn+k}{2}\right)}\\
&= \frac{k}{\Gamma(\frac{k+2}{2})m!}\frac{2^{n+1}\pi^{(m-1)/2}\Gamma(\frac{n-m+2}{2})\Gamma(\frac{nm+n-m+1}{2})}{\Gamma(\frac{n+1}{2})^m\Gamma(n-m+1)}\\
&\qquad \Gamma\left(\frac{n+2}{2}\right)^{(mn+k)/n}\frac{\Gamma\left(\frac{mn+k}{n}\right)\Gamma\left(\frac{k}{2}\right)}{2n\Gamma\left(\frac{mn+k}{2}\right)}\\
&= \frac{1}{m!}\frac{2^{n+1}\pi^{(m-1)/2}\Gamma(\frac{n-m+2}{2})\Gamma(\frac{nm+n-m+1}{2})}{\Gamma(\frac{n+1}{2})^m\Gamma(n-m+1)}\\
&\qquad \Gamma\left(\frac{n+2}{2}\right)^{(mn+k)/n}\frac{\Gamma\left(\frac{mn+k}{n}\right)}{n\Gamma\left(\frac{mn+k}{2}\right)}\\
&= \frac{1}{m!}\frac{2^{m+1}\pi^{m/2}\Gamma(\frac{nm+n-m+1}{2})}{\Gamma(\frac{n+1}{2})^m\Gamma(\frac{n-m+1}{2})}\\
&\qquad \Gamma\left(\frac{n+2}{2}\right)^{(mn+k)/n}\frac{\Gamma\left(\frac{mn+k}{n}\right)}{n\Gamma\left(\frac{mn+k}{2}\right)} \qquad (2.22)
\end{aligned}
$$

It follows from (2.5), (2.8), (2.17), and (2.18) that, with $m = n-k$ still the codimension,

$$
E(n,k) = \frac{2^{m+1}\pi^{m/2}\Gamma\left(\frac{n+2}{2}\right)^{m+k/n}\Gamma\left(\frac{mn+n-m}{n}\right)\Gamma\left(\frac{mn+n-m+1}{2}\right)}{n\Gamma(m+1)\Gamma\left(\frac{n-m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)^m\Gamma\left(\frac{mn+n-m}{2}\right)}. \qquad (2.23)
$$

Each point of the k-skeleton is in $n-k+1$ cells, so the expected density of the k-skeleton per unit n-volume is

$$
\rho(n,k) = \frac{E(n,k)}{m+1}. \qquad (2.24)
$$

3 Cross-sections

Consider a random q-dimensional cross section of a random n-dimensional tessellation. The density $\rho(n,p,q)$ of the p-dimensional skeleton of the cross-section will be proportional to the density of the $(n+p-q)$-dimensional skeleton of the tessellation:

$$
\rho(n,p,q) = \rho(n,n+p-q)\cdot K(n,q,n+p-q), \qquad (3.1)
$$

where $K(n,q,k)$ is the expected density of $q+k-n$ dimensional measure of the intersection of random unit density k-planes with a q-plane in n-space:

$$
K(n,q,k) = \frac{\Gamma\left(\frac{q+1}{2}\right)\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{q+k-n+1}{2}\right)}. \qquad (3.2)
$$

With m still the codimension, $m = q - p$,

$$\rho(n,p,q) = \frac{2^{m+1}\pi^{m/2}\Gamma\left(\frac{n+2}{2}\right)^{m+1-m/n}\Gamma\left(\frac{mn+n-m}{n}\right)\Gamma\left(\frac{mn+n-m+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{n\Gamma(m+2)\Gamma\left(\frac{n+1}{2}\right)^{m+1}\Gamma\left(\frac{nm+n-m}{2}\right)\Gamma\left(\frac{p+1}{2}\right)}. \tag{3.3}$$

One quantity of interest not derived above is the number density $D(n, q)$ of cells in a cross section. In general, numbers of features are not expressible as measures and so are not accessible to the techniques used here. But for $q = 1$, the number density of cells is the same as the number density of vertices, so

$$D(n,1) = \rho(n,0,1). \tag{3.4}$$

and for $q = 2$, by Euler's formula the cell density is half the vertex density, so

$$D(n,2) = \frac{\rho(n,0,2)}{2}. \tag{3.5}$$

4 Asymptotic values

There are some interesting asymptotic values as the dimension n approaches infinity. The number of vertices per cell is asymptotically

$$E(n,0) \approx 2^{(n+1)/2}e^{1/4}\pi^{(n-1)/2}n^{n/2-1}, \tag{4.1}$$

which grows rather rapidly.

The ratio of the measure of the boundary of a cell to that of a ball of unit volume has a limit:

$$\lim_{n\to\infty}\frac{E(n,n-1)}{n\alpha(n)^{1/n}} = \sqrt{2}, \tag{4.2}$$

so cells do not become wildly unspherical as the dimension increases.

The average edge length asymptotically is

$$\frac{2E(n,1)}{nE(n,0)} \approx \frac{1}{n\sqrt{e}}. \tag{4.3}$$

Most interesting are asymptotic values for cross-sections. For fixed p and q, there is a finite asymptotic value:

$$\rho(\infty,p,q) = \lim_{n\to\infty}\rho(n,p,q) = \frac{2^m\pi^{m/2}e^{m/2}\Gamma\left(\frac{q+1}{2}\right)}{(m+1)^{1/2}\Gamma\left(\frac{p+1}{2}\right)}. \tag{4.4}$$

5 Infinite dimensional tessellations

The finiteness of the limit in (4.4) would seem to imply that we have cross-sections of tessellations of infinite dimensional space. The definition of a Voronoi tessellation in an infinite dimensional space runs into severe problems. It is natural to try to apply the finite dimensional definition to Hilbert space. Unfortunately, a ball of finite radius in Hilbert space has zero volume, so Hilbert space itself, as a union of finite radius balls, has zero volume. Hence a unit density Poisson process produces no points and no tessellation.

R^∞ does have infinite volume, and so it can support a nontrivial Poisson point process. But the Euclidean distance between almost all pairs of points is infinite, so the concept of nearest neighbor becomes rather meaningless. Alternately, one could define the cell of a point of S as bounded by all hyperplanes that are perpendicular bisectors of the segments between the point and the other points of S. But the notion of perpendicularity is also unavailable. Only the affine properties of R^∞ are useful, but the definition of Voronoi tessellation requires metric properties.

6 Random tessellation processes

It may not be possible to define a Voronoi tessellation of infinite dimensional space, but it is in a sense possible to strictly define a cross-section of a random infinite dimensional space.

Define a *random tessellation process* on R^n to be a stochastic method of producing tessellations of R^n. Strictly speaking, it is a probability measure on the suitably defined set of all tessellations of R^n.

Define $\mathrm{V}(n)$ to be the random Voronoi tessellation process on R^n, with probability measure on tessellations induced from the probability measure defining the Poisson point process. This process was the subject of section 2.

Define $\mathrm{X}(n,q)$ to be the cross section tessellation process induced by $\mathrm{V}(n)$ on R^q by the canonical embedding of R^q in R^n. This process was the subject of section 3.

The cross section process $\mathrm{X}(n,q)$ may be defined in terms of a process involving only R^{q+1}. In the original definition, the type of a point T in R^q depends on the number of nearest neighbors in S, which means that a sufficient set of information is the set of distances from points in R^q to points in S. For this, it is sufficient to know for each point of S the projection on R^q and the distance from R^q. Map $\mathrm{R}^n = \mathrm{R}^q \times \mathrm{R}^{n-q}$ to $W = \mathrm{R}^q \times [0,\infty)$ by

$$(x,y) \mapsto (x,w), w = \alpha(n-q)||y||^{n-q}. \tag{6.1}$$

Note that a unit density Poisson point process on R^n induces a unit density Poisson point process on W. For nearest neighbor calculations, any monotone function of distance will serve. It will be convenient to use the volume of the ball whose radius is the distance concerned. Let x_0 be a point in R^q and (x,y) a point in S that maps to (x,w) in W. Then the volume $V_n(x_0,x,w)$ of the ball centered at $(x_0,0)$ with $(x_1 y)$ on its boundary is

$$V_n(x_0,x,w) = \alpha(n)\left((x-x_0)^2 + \left(\frac{w}{\alpha(n-q)}\right)^{2/(n-q)}\right)^{n/2} \tag{6.2}$$

Hence the tessellation process on R^q can be defined entirely in terms of a unit density point process on W. The limiting process $\mathrm{X}(\infty,q)$ can then be defined using the neighbor distance function

$$V_\infty(x_0,x,w) = \lim_{n\to\infty} V_n(x_0,x,w) = e^{-q/2} w e^{\pi e(x-x_0)^2}. \tag{6.3}$$

Each cell in R^q is still formed as the intersection of half-spaces, and so is a convex polyhedron.

7 Simulation algorithm

A sample $\mathrm{X}(\infty,q)$ tessellation can be efficiently generated with an insertion algorithm that is a modification of the one in Bowyer [4]. The modified algorithm is briefiy outlined here. The tessellation is represented with some data structure that is updated as new seeds are generated. Use the data structure of your choice to represent the tessellation, but each vertex should have its nearest neighbor V_∞ distance stored with it.

Initialization: $q+1$ seeds giving a tessellation with a vertex.
Repeat as many times as desired:
    Generate the next random seed in W.
    Find all old vertices for which new seed is within their V_∞ distances.
        These vertices are to be eliminated from the new configuration.
        If none, continue to next seed.
    On each edge connecting an eliminated vertex and a remaining vertex, calculate a new vertex.
    Add a new cell with the new vertices to the data structure.

Figure 1 shows a sample tessellation of the plane.

Appendix. Notation.

n	dimension of tessellated space.
k	dimension of skeleton.
q	dimension of cross-section.
p	dimension of skeleton in cross-section.
m	codimension of skeleton, $m = n - k = q - p$.
S	set of points generating the tessellation.
T	point of the k-skeleton of the tessellation.
S_i	a nearest neighbor seed of T.
$\Gamma(n)$	gamma function.
$\alpha(n)$	measure of n-dimensional ball, $\alpha(n) = \pi^{n/2}/\Gamma(1 + n/2)$.
$\alpha'(n)$	measure of boundary of n-ball, $\alpha'(n) = n\alpha(n)$.
$E(n, k)$	expected measure of k-skeleton per cell in n-space.
$\rho(n, k)$	expected density of k-skeleton in n-space.
$\rho(n, p, q)$	expected density of p-skeleton in q-dimensional cross-section of n-space.
$\rho(\infty, p, q)$	limiting value of $\rho(n, p, q)$ as $n\infty$.
$D(n, q)$	number density of cells in q-dimensional cross-section of, n-dimensional space.
$\mathrm{V}(n)$	random Voronoi tessellation process in R^n.
$\mathrm{X}(n, q)$	q-dimensional tessellation process, cross-section of $\mathrm{V}(n)$.
$\mathrm{X}(\infty, q)$	limiting tessellation process.
W	alternate seed domain for cross-section processes, $W = \mathrm{R}^q \times [0, \infty)$.
x_0	point in R^q.
(x, w)	point in W.
$V_n(x_0, x, w)$	n-ball volume, used as distance function.
$V_\infty(x_0, x, w)$	distance function for limit tessellation process.

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