# Minimal Surfaces, Corners, and Wires

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**Abstract.**

Weierstrass representations are given for minimal surfaces that have free boundaries on two planes that meet at an arbitrary dihedral angle. The contact angles of a surface on the planes may be different. These surfaces illustrate the behavior of soapfilms in convex and nonconvex corners. They can also be used to show how a boundary wire can penetrate a soapfilm with a free end, as in the overhand knot surface. They should also cast light on the behavior of capillary surfaces.


Keywords: minimal surfaces, Weierstrass representation, capillary surfaces, boundary values.

AMS Subject Classification(1990): primary 49Q05; secondary 53A10, 35J67.

## 1. Introduction.

The shapes of soap films have been studied by mathematicians for centuries (see [NJ] for an extensive summary). But not everything is yet known. F. Almgren once tried to prove that the boundary wire of a soap film could have no free ends, in line with the pervasive mathematical principle that a boundary has no boundary. He failed by finding a counterexample: the soap film that forms on a wire in the shape of an open overhand knot. What exactly happens at the points where the boundary wire leaves the soapfilm? A physical wire has a nonzero diameter, so the boundary of the soapfilm is really on a tubular surface, not a curve. Given that three surfaces must meet at angles of $2\pi/3$ and that the soapfilm must meet the wire surface at right angles, one can figure out that the soapfilm looks like figure 1. A triple line comes in from the upper right and curves to hit the wire perpendicularly. Above the triple line is a more or less flat vertical "wall" with its left edge on the wire. Below the triple line, the surface forms a "skirt" around the wire.
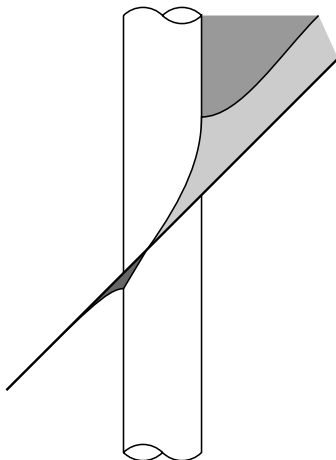


**Figure 1.** Side view of the free end of thick wire boundary going through soapfilm. Note that all surfaces and the triple line meet the wire at right angles.

Is the thickness of the wire essential? In 1976, Almgren stated in [AT] that in reference to the overhand knot, "No mathematical soapfilm-like configuration can form on a similar, infinitely thin frame that has two free ends." He later realized that one could, but did not find the correct configuration.

In my own investigation of the Opaque Cube Problem [BK1], I came across similar configurations of boundary wires going through surfaces. I set out to see what happened in the case of the boundary wire being a true zero diameter line. I expected that for a tame boundary, the surface would be made up of tamely curving surfaces with tamely curving triple lines and some isolated tame point singularities. Probably the wire would go through the surface with some kind of cusp. But by contemplating the general properties of Weierstrass representations for minimal surfaces satisfying the appropriate boundary conditions, I realized that the "skirt" has to run along the wire for a short distance. I used a program of mine, called the Surface Evolver [BK2], to compute an example to verify my calculations, and found that I had to magnify by a factor of 100,000 to see

the detail! Figure 2 shows this close-up view. This disparity in scale with the boundary wire specifications is amazing. Past results on minimal surfaces have always been on the theme that the surface is at least as smooth as the boundary. My results don't technically contradict that, but they were certainly unexpected.
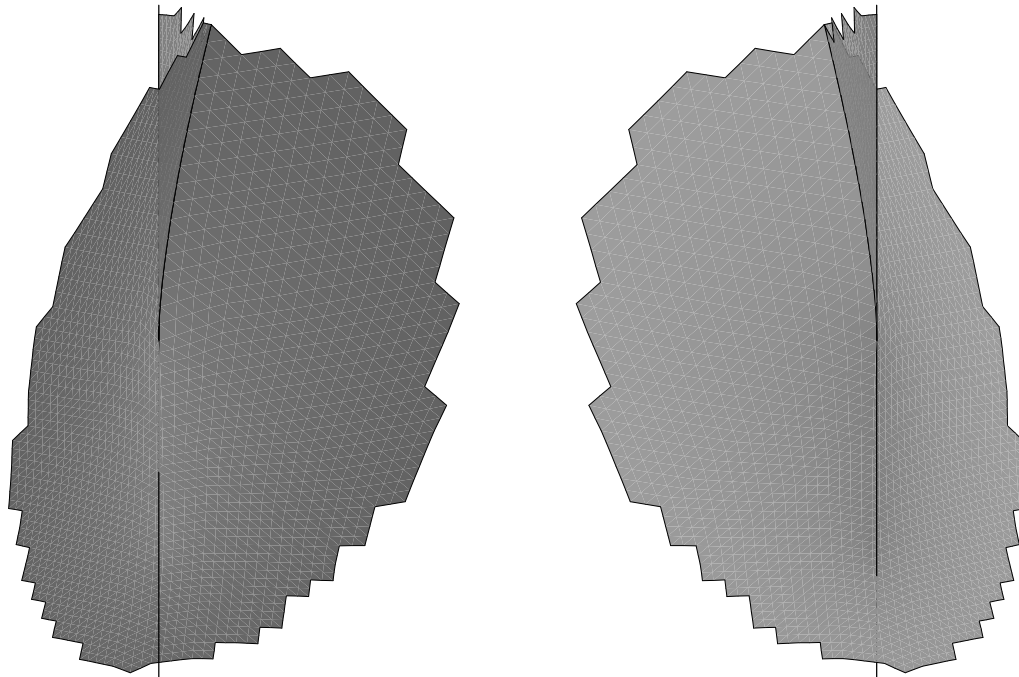


**Figure 2.** 100,000 fold magnification of a zero diameter boundary wire leaving a soapfilm. Slightly oblique view. From my Surface Evolver program.

This paper contains the results of my investigations of the problem and its generalization. The general problem I study here is that of a minimal surface that has part of its boundary on two half planes meeting in a corner at an arbitrary dihedral angle. The surface meets each plane at a fixed contact angle, which may be different on the two planes. I find the general forms of the Weierstrass representations at critical points on the corner, and some complete sample solutions. Some numerical results are given in case E below. The thin wire free end problem can be subsumed in these results by viewing the "wall" part of the surface as part of a half-plane whose sides form a corner with interior angle of $2\pi$ with the "skirt" being a surface with contact angle $\pi/3$ on the wall.

Much previous work on minimal surfaces in corners has been focused on the Dirichlet problem of a given boundary. Beeson [BM] treated the case of a corner formed of two straight lines. Elcrat and Lancaster [EL1,EL2] investigated surfaces over a non-convex quadrilateral in the non-parametric case, which is equivalent to having walls above the sides of the quadrilateral.

For the Neumann problem, previous work on surface shapes in corners has been done for constant (or at least continuous) contact angle $\gamma$ for capillary surfaces viewed as graphs of functions.Minimal surfaces wrapping around the edge of a half-plane with contact angle

$\pi/2$ on both sides have been studied in [HN]. In the case $\pi - 2\gamma < \alpha < \pi$, Simon [SL] showed the function for the surface is of class $C^1$, and Tam [TL] showed additionally that for $\alpha = \pi - 2\gamma$ the normal vector is continuous. If $\alpha = \pi$, then there is just a featureless flat wall. Korevaar [KN] showed that if $\alpha > \pi$ then there exist surfaces whose functions are discontinuous at the corner. Lancaster [LK] derived the possible behaviors of radial limits at a non-convex corner. Examples of all of these types of behavior will be seen in this paper.

## 2. Preliminaries.

First, we will set up the geometry of a corner. Let the two half-planes of the wall be designated $A$ and $B$, and let the corner line $L$ be their common edge. Let $\alpha$ be the angle between the planes, measured on the side with the surface. The range of $\alpha$ is $0 < \alpha \leq 2\pi$. The upper limit is physical, not mathematical. If one wants to visualize space as wrapping around $L$ indefinitely (i.e. a simply connected covering space of the complement of $L$), then all the analysis in this paper will still apply.

The curve where the surface meets the wall is the *trace* of the surface. In order for the notion of contact angle to make sense, one side of the surface will be designated the *outside* and the other side the *inside*. The contact angles $\gamma_A$ and $\gamma_B$ are measured from the inside of the surface to the wall. Their range is $0 < \gamma_A, \gamma_B < \pi$. Without loss of generality, we will take $\gamma_A \geq \gamma_B$. For contact angles of $0$ or $\pi$ there can be no trace on a half-plane, but there may be a trace along the corner.

Physically, a contact angle arises when the surface is the boundary between two fluids (air and water, for example) which wet the wall with different surface energies. The cosine of the contact angle is the ratio of the difference in wall surface energies to the surface tension of the surface. Thus the problem in this paper corresponds to two different materials and two different fluids meeting at a corner. That the surface is a minimal surface implies that there are no external forces such as gravity, and that there are no constraints such as fixed volumes.

A minimal surface can be parameterized with a complex variable by the Weierstrass representation. Let $f(u)$ and $g(u)$ be any meromorphic functions of a complex variable $u$. (My notation mostly follows [OR].) Then a parameterization of a minimal surface in $\mathbf{R}^3$ is given by the real parts of complex integrals:

$$x = \Re \int (1 - g^2) f \, du,$$
$$y = \Re \int i(1 + g^2) f \, du, \tag{1}$$
$$z = \Re \int 2gf \, du.$$

The value of $g$ can be interpreted as the normal to the surface via the Gauss map, which identifies a complex value with a unit vector by stereographic projection. In particular, the normal vector is

$$\vec{N} = \left( \frac{2\Re g}{|g|^2 + 1}, \frac{2\Im g}{|g|^2 + 1}, \frac{|g|^2 - 1}{|g|^2 + 1} \right). \tag{2}$$

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The sphere of unit normals will be referred to as the *Gauss sphere,* and the image of the surface under the Gauss map will be its *Gauss image.* The function $f$ is a conformal factor. If the surface is oriented so that $g = 0$ at a point, then $dx - idy = f du$ at the point, so for $f(0)$ positive real, the positive real $u$ axis maps to the positive $x$ axis and the positive imaginary $u$ axis maps to the negative $y$ axis. If $g = \infty$ at a point, then $dx + idy = -g^2 f du$ there, so $f$ needs to have a root of high enough order to tame the poles of $g$. The orientation of the mapping for other $g$ can be found by parallel translating the tangent plane of the Gauss sphere along a meridian from the south pole to $g$.

In this paper, $f$ and $g$ may be functions of various parameters. The same function names will be used in all cases. Thus if $u$ and $w$ are different parameters, $g(u)$ and $g(w)$ are technically different functions, but have the same value when $u$ and $w$ correspond to the same point on the surface. Since $f(u)du$ is a differential, the proper relation is $f(u)du = f(w)dw$, or $f(u) = f(w)(dw/du)$. Where there is any ambiguity, the parameter will be explicit. Often the parameter will be $g$ itself.

The traces on each half-plane map to the Gauss sphere on circles of central angles $\gamma_A$ and $\gamma_B$ around the normal vectors of the planes. These will be referred to as the *Gauss circles* of the half-planes. When the trace goes along the line $L$, the Gauss map goes along a great circle perpendicular to $L$, which will be the Gauss circle for $L$. Not all of the Gauss circle for $L$ represents stable orientations. Some orientations are unstable against perturbations onto the half-planes.

The condition on $f(u)$ needed to guarantee that the trace be on a half-plane is that the trace tangent vector be perpendicular to the normal vector. If $\vec{N} = (N_x, N_y, N_z)$ is the normal and $\vec{dx}$ is the trace tangent in terms of differentials, then $\vec{N} \cdot \vec{dx} = 0$, or

$$\Re \left[ \left( (1 - g^2)N_x + i(1 + g^2)N_y + 2gN_z \right) f du \right] = 0. \tag{3}$$

This will be referred to as the *trace condition* for the surface on a half-plane. A special case will be used often below:

**Lemma 1.** *If the Gauss circle of a plane passes through the north pole of the Gauss sphere and $g_0$ is any point on the Gauss circle, then the trace condition (3) is satisfied if and only if $\Im[(g - g_0)^2 f(g)] = 0$ for all $g$ on the Gauss image of the trace.*

*Proof.* Let $\gamma$ be the contact angle and let $\theta$ be the argument of the plane normal $\vec{N}$, so that

$$\vec{N} = (\sin \gamma \cos \theta, \sin \gamma \sin \theta, \cos \gamma).$$

Then the trace condition (3) becomes

$$\Re \left[ \left( (1 - g^2) \sin \gamma \cos \theta + i(1 + g^2) \sin \gamma \sin \theta + 2g \cos \gamma \right) f(g)dg \right] = 0,$$

or

$$\Re \left[ \left( \sin \gamma (e^{i\theta} - e^{-i\theta} g^2) + 2 \cos \gamma \right) f(g)dg \right] = 0.$$

In the $g$ plane, the Gauss circle becomes a straight line through $g = e^{i\theta} \cot \gamma$ with direction angle $\theta + \pi$. Parameterize the line as $g = e^{i\theta}(\cot \gamma + iv)$, so $dg = ie^{i\theta}dv$. Plugging in, simplifying, and dropping real factors leaves

$$\Im \left[ e^{2i\theta} f(g) \right] = 0,$$

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which, for $g$ and $g_0$ on the line, is equivalent to

$$\Im \left[ (g - g_0)^2 f(g) \right] = 0. \qquad\blacksquare$$

The strip of surface next to the trace maps to a strip in the Gauss sphere. One can get a rule to see which side of the Gauss trace the image is on. Since the surface is a minimal surface, its sectional curvatures are of opposite sign at any point. Hence if the surface is on the convex side of the trace (for $\gamma \le \pi/2$), the Gauss image is outside the circle, and vice versa. For $\gamma > \pi/2$, the concave side of the trace is mapped to the inside of the Gauss circle (the inside being the side where the half-plane normal is).

The problem of the shape of the surface at the corner can be conveniently viewed in terms of the Gauss map: Given the contact angles and the angle between the planes, how does the trace get from the Gauss circle for $A$ to that of $B$? In general, the trace can wander around on the circles for $A$, $B$, and $L$, doubling back on itself, and being generally complicated. There are many ways to get from one circle to another, but only some of them give traces on the proper sides of the half-planes. Other choices than those discussed below lead to traces on the extensions of the half-planes beyond $L$ or on the back sides of the planes. In this paper, I will not try to discuss all possible behaviors, but I will discuss representative ones.

The bulk of this paper considers various cases categorized by the angle $\alpha$ between the walls. Bear in mind that one angle may belong to several cases, and which case applies will depend on the boundary conditions for the rest of the surface away from the corner. Also each case may have variations differing in orientation.

The procedure for finding explicit representations starts first with choosing the trace path and Gauss image in the Gauss sphere. Second, a convenient parameterization domain is chosen. Often this can be the Gauss image itself. Third, knowing $g$ on the domain, it remains to find $f$. We know $g$ and $\arg(du)$, so the trace condition (3) boils down to a constraint on $\arg(f)$. That is, $\Im \log(f)$ is given on part of the boundary of the domain. Likewise, when the trace lies along $L$, $\vec{dx}$ is automatically perpendicular to the surface normal $\vec{N}$, so the only additional condition needed is that $\vec{dx}$ be perpendicular to some other convenient vector, which again leads to a prescribed value of $\Im \log(f)$ on part of the domain boundary. $\Im \log(f)$ can be extended many ways to a harmonic function on the domain, which then leads to the complex function $\log(f)$ and then to $f$ itself. So the existence of surfaces is not a problem, but actually finding explicit representations may be.

The general form of the Weierstrass representation at a critical point $P$ on the trace on the corner will be standardly parameterized by a complex parameter $w$ in the upper $w$ half-plane. The strategy for finding the general form at $P$ will be: First, orient the corner so that the Gauss circles involved are in convenient positions. Second, figure out the image of the trace and surface in the $g$ plane. In general, the point $P$ will correspond to a $g_0$ at the intersection of two Gauss circles. Third, find a formula $g(w)$ mapping the upper $w$ half-plane to the Gauss image with $g(0) = g_0$. The Gauss image may wrap arbitrarily many times around $g_0$, which introduces arbitrary factors of $w$ into $g(w)$. Fourth, use the trace condition via Lemma 1 or an equivalent argument to get a reality condition on $f(w)$ on the real $w$ axis. Fifth, use the angle subtended by the surface around $P$ to determine

the leading power of $f(w)$. Finally, write $f(w)$ in terms of an arbitrary function $h(w)$ that is analytic at $w = 0$, is real on the real $w$ axis, and has $h(0)$ positive.

To study the smoothness of the surface and trace at the point, we will change from parameter $w$ to a parameter $\zeta$ such that $f(\zeta)$ is nonzero at $P$. Then the leading terms in the asymptotic expansion of $g(\zeta)$ will reveal the order of continuity of the normal vector at $P$. The surface and trace will have order of continuity one higher. The Gauss map $g(w)$ will include arbitrary factors of $w$, and the more factors, the smoother the surface. The extra factors are not the general case, however.

## 3. Case A: $\alpha > \pi + \gamma_A + \gamma_B$.

This case is illustrated by figure 3. The corner angle is large enough that the Gauss circles of the half-planes are disjoint. In this type of surface, the Gauss map of the trace goes along the Gauss circle of $L$ to get from one circle to the other. This means that the trace on the wall (moving from left to right in figure 3) curves down to become tangent to $L$ at $Q$, runs down $L$ a ways to $R$, turns around and runs back up to $S$, and finally curves off toward $T$. The trip along $L$ is necessary to let the surface normal rotate, since the half-planes can have no common surface normal. It is this case with $\alpha = 2\pi$ and $\gamma_A = \gamma_B = \pi/3$ that corresponds to the free end of a wire coming out of a soapfilm, shown in figure 2.
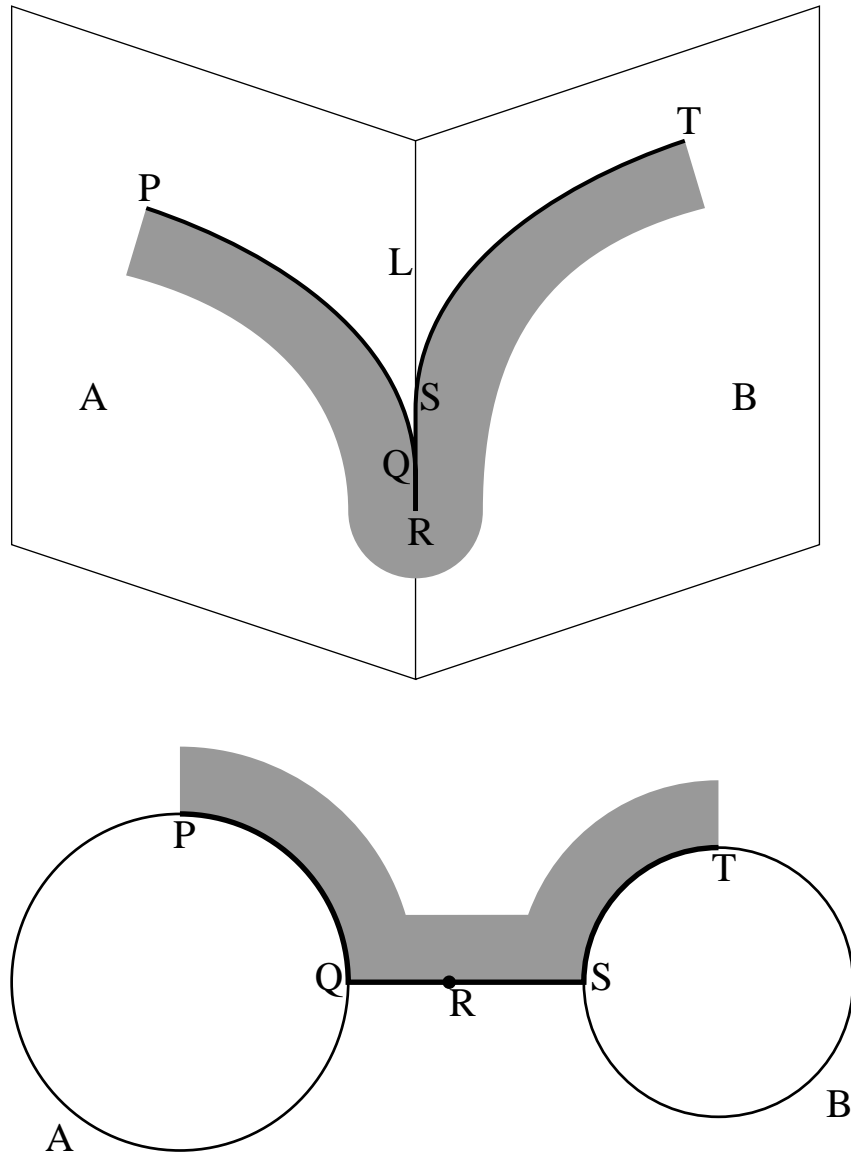


**Figure 3.** Case A: $\alpha > \pi + \gamma_A + \gamma_B$. Upper, surface trace. The trace runs along the corner from Q down to R and up to S. Lower, Gauss map.

To find the general form of the Weierstrass representation at $R$, orient coordinates so that $g_R = 0$ and corner $L$ lies along the $y$ axis, making the Gauss circle for $L$ lie on the real axis of the $g$ plane. The trace condition for $f(g)$ on $L$ can be taken to be $dz = 0$ or $\Re[gf(g)dg] = 0$ on the real $g$ axis, or $\Re[f(g)] = 0$. Take the Gauss map in the upper $w$ half-plane to be $g(w) = w^{1+2k}$, $k \geq 0$, which wraps the Gauss image $k$ times around $g_R$. Recalling that $f(g) = f(w)/(dg/dw)$, we get $\Re f(w) = 0$ on the real $w$ axis. Hence $f(w)$ is analytic at $w = 0$. Because the surface subtends angle $2\pi$ at $R$, the leading power of $f(w)$ must be $w$. For the trace to run to the left of $R$, the leading coefficient must be positive imaginary. Thus the general Weierstrass representation is

$$g(w) = w^{1+2k}, \qquad f(w) = iwh(w),$$

where (as always in this paper) $h(w)$ is real-valued on the real axis with $h(0)$ positive. For smoothness, take $\zeta = w^2$, so that $g(\zeta) = \zeta^{1/2+k}$. Thus the surface normal is of class $C^{k,1/2}$ and the surface and trace are of class $C^{k+1,1/2}$. The surface curves away from the corner on the opposite side of $R$ from the trace like

$$z \propto |y|^{3/2+k}.$$

To find the general form at $Q$ (or at $S$), orient the corner so that the Gauss circle for $A$ passes through the north pole of the Gauss sphere and is in the $x \geq 0$ half-space, and $L$ again lies on the $y$ axis. From the $z$ component of formula (2) it follows that $g_Q = \cot\gamma_A$. In the $g$ plane, the Gauss image is bounded above by the real axis and on the right by the line $\Re g = g_Q$. The trace condition on $f(g)$ is again $\Re f(g) = 0$ on the real axis and, by Lemma 1, $\Im\left[(g - g_Q)^2 f(g)\right] = 0$ for $\Re g = g_Q$, or $\Im f(g) = 0$. The general form for $g(w)$ on the upper $w$ half-plane is $g(w) = g_Q - w^{1/2+2k}$. Hence $\Re f(w) = 0$ on both halves of the $w$ axis. Since the surface subtends angle $\pi$ at $Q$, the leading term of $f(w)$ is a constant. Thus the general Weierstrass representation is

$$g(w) = \cot\gamma - w^{1/2+2k}, \qquad f(w) = ih(w).$$

For smoothness, take $\zeta = w$, so that $g(\zeta) - g_Q = -\zeta^{1/2+2k}$. Thus the surface normal is of class $C^{2k,1/2}$ and the surface and trace are of class $C^{k+1,1/2}$. The general shape of the trace on the half-plane as it approaches $Q$ is

$$|x| \propto |z| \propto |y|^{3/2+2k}.$$

Exact solutions for the entire corner can be found for $\alpha = 2\pi$, which corresponds to the surface wrapping around the edge of a half-plane. To get a nice domain, orient the half-plane so that it is the $z = 0$, $x \leq 0$ half-plane with side $A$ on top and side $B$ on the bottom. In the $g$ plane the domain is in an annulus between the Gauss circles. The radii of the circles can be calculated from the values of $g$ at points $Q$ and $S$ using (2):

$$\frac{g_Q^2 - 1}{g_Q^2 + 1} = \cos\gamma_A, \qquad \frac{g_S^2 - 1}{g_S^2 + 1} = -\cos\gamma_B,$$

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so

$$g_Q = \cot(\gamma_A/2), \qquad g_S = \tan(\gamma_B/2).$$

For the parameter, take $w = \log g$. Then the domain in the $w$-plane is a half-strip. The trace condition for the entire trace may be taken to be $dz = 0$ or $\Re[2g(w)f(w)dw] = 0$. Since $g(w) = e^w$, this means that on the left and right edges of the domain (where $dw$ is imaginary) we have $\Im[e^w f(w)] = 0$. On the lower edge $dw$ is real, so $\Re[e^w f(w)] = 0$. Hence $e^w f(w)$ must have zeroes at $Q$ and $S$ (poles would send the surface off to infinity). Furthermore, to get the turnaround at $R$, there must be a zero at some $w_R$ between $Q$ and $S$. By the reflection principle, $e^w f(w)$ must be periodic in the real direction. One way to do all this is

$$f(w) = Ce^{-w}\sin\left(2\pi\frac{w - w_S}{w_Q - w_S}\right),$$

where $C$ is an arbitrary real constant which just sets the scale of the surface.

If $e^w f(w)$ is real on the vertical line through $w_R$ (as it is in the above solution), then $dz = 0$ on this line and it maps to the $x > 0$ extension of the half-plane, and the angle the surface makes with the extension is constant. Thus the solution can be decomposed into two case E solutions with $\alpha = \pi$.

This case illustrates one of the two possible behaviors that Lancaster [L] derived for radial limits at a non-convex corner for a discontinuous trace. If in figure 3 the $x, y$ plane is endowed with polar coordinates $r, \theta$ with origin at the corner, plane $A$ at $\theta = 0$, and plane $B$ at $\theta = \alpha$, and the surface is represented as $z = f(r, \theta)$, then the *radial limit* $Rf(\theta)$ is

$$Rf(\theta) = \lim_{r \to 0^+} f(r, \theta).$$

The behavior according to [L] is that the radial limit is constant on an interval of $\theta$, then monotone decreasing, then constant for interval $\pi$, then monotone increasing, then finally constant. Here there is a $\theta_0$ such that

$$Rf(\theta) = \begin{cases} z_Q, & \text{for } 0 \leq \theta \leq \gamma_A, \\ \text{decreasing}, & \text{for } \gamma_A < \theta < \theta_0, \\ z_R, & \text{for } \theta_0 \leq \theta \leq \theta_0 + \pi, \\ \text{increasing}, & \text{for } \theta_0 + \pi < \theta < \alpha - \gamma_B, \\ z_S & \text{for } \alpha - \gamma_B \leq \theta \leq \alpha. \end{cases}$$

The direction of $\theta_0$ is perpendicular to $L$ and $g_R$.

**4. Case B:** $\alpha = \pi + \gamma_A + \gamma_B$.

This case is illustrated by figure 4. The two planes have exactly one possible common surface normal and the Gauss circles meet tangentially. In this type of surface, the Gauss map of the trace goes from one circle to the other where they contact. This means that the trace on the wall (moving from left to right) curves down to become tangent to $L$ at $R$, then turns around and runs back up, curving off toward $T$. The trace does not run along $L$ for any distance.
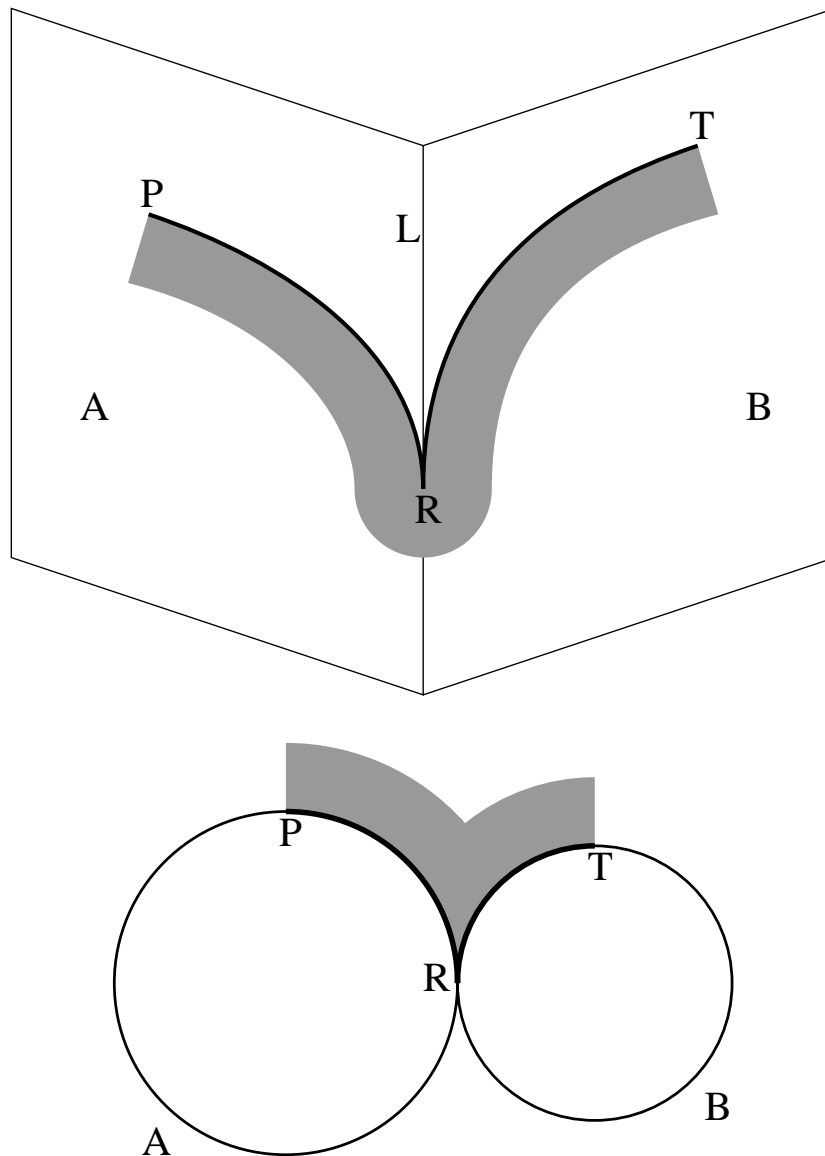


**Figure 4.** $\alpha = \pi + \gamma_A + \gamma_B$. Upper, surface trace. The trace is tangent to the corner where it crosses, but it does not run along the corner. Lower, Gauss map.

The only critical point is $R$. To study the surface around $R$, orient the corner so that $L$ is the $y$-axis and the common normal (at $R$) points vertically upward and half-plane $A$

is in the positive $x$ half-space. Then the Gauss circles pass through the north pole of the Gauss sphere and the domain in the $g$ plane is the strip $-\cot\gamma_B \leq \Re g \leq \cot\gamma_A$, with $g_R = +i\infty$. By Lemma 1, the trace condition on the line $\Re g = \cot\gamma_A$ can be taken to be $\Im[(g - \cot\gamma_A)^2 f(g)] = 0$, which is equivalent to $\Im f(g) = 0$. Likewise, $\Im f(g) = 0$ on the line $\Re g = -\cot\gamma_B$. In the upper $w$ half-plane, define the Gauss map to be

$$g(w) = iw^{-2k} + \frac{\cot\gamma_A + \cot\gamma_B}{i\pi}\log w - \cot\gamma_B.$$

This wraps the Gauss image $k$ times around the north pole with the positive $w$ axis mapping to the $A$ trace and the negative $w$ axis mapping to the $B$ trace. The trace conditions for $f$ on the real $w$ axis become

$$\Im f(g) = \Im\left[f(w)\frac{dw}{dg}\right] = \Im\left[f(w)\frac{i\pi w}{\cot\gamma_A + \cot\gamma_B}\right] = 0,$$

which reduces to $\Re f(w) = 0$. The surface at $R$ subtends an angle of $2\pi$, so

$$g(w)^2 f(w)dw \propto wdw,$$

which means $f(w)$ has a zero of order $4k+1$. Hence the general Weierstrass representation is

$$g(w) = iw^{-2k} + \frac{\cot\gamma_A + \cot\gamma_B}{i\pi}\log w - \cot\gamma_B,$$
$$f(w) = iw^{4k+1}h(w).$$

For smoothness, let $\zeta = w^2$. To avoid awkwardness in judging continuity involving infinities, look at the continuity of $1/g$, which will have the same continuity as the surface normal. Then

$$1/g = \left[i\zeta^{-k} + \frac{\cot\gamma_A + \cot\gamma_B}{2i\pi}\log\zeta - \cot\gamma_B\right]^{-1},$$

or, essentially,

$$1/g \approx \zeta^k[1 + a\zeta^k\log\zeta + b\zeta^k]^{-1}.$$

Hence

$$1/g \approx \begin{cases} 1/(1 + a\log\zeta + b), & \text{if } k = 0; \\ \zeta^k - a\zeta^{2k}\log\zeta - b\zeta^{2k} + \ldots, & \text{if } k > 0. \end{cases}$$

Thus the normal is of class $C^0$ or almost of class $C^{2k}$ respectively.

Exact solutions can be found for arbitrary contact angles. For sample solutions, it is convenient to use the strip in the $g$ plane as the domain. One function that works is

$$f(g) = \sec\left(2\pi\frac{g - \cot\gamma_B}{\cot\gamma_A - \cot\gamma_B}\right).$$

This dies off at $g = \infty$ fast enough to keep the surface finite. The factor 2 is included to get the traces to go in the right direction. The shape of the trace on the half-planes works out to be essentially

$$|x| \propto |z| \propto \left|\frac{y}{log|y|}\right|.$$

**5. Case C:** $\pi + \gamma_A - \gamma_B < \alpha < \pi + \gamma_A + \gamma_B$.

Here there is a common normal $\vec{N}$ to both half-planes, the Gauss circles cross transversely, and the trace reaches the corner obliquely at a point $R$. There are several subcases, as shown in figures 5, 6, and 7, depending on the concavity of the trace.

To find the general Weierstrass representation at $R$, orient the corner so that the intersection of the Gauss circles that is not the $R$ intersection is at the north pole and the center of the circle for $A$ is on the positive real axis. Let $\theta$ be the argument of $g_B$, which can be found by a little spherical trigonometry:

$$\cos \theta = \frac{\cos(\alpha - \pi) - \cos \gamma_A \cos \gamma_B}{\sin \gamma_A \sin \gamma_B}.$$

Note that the angle subtended by the surface around $R$ is $\theta + \pi$.
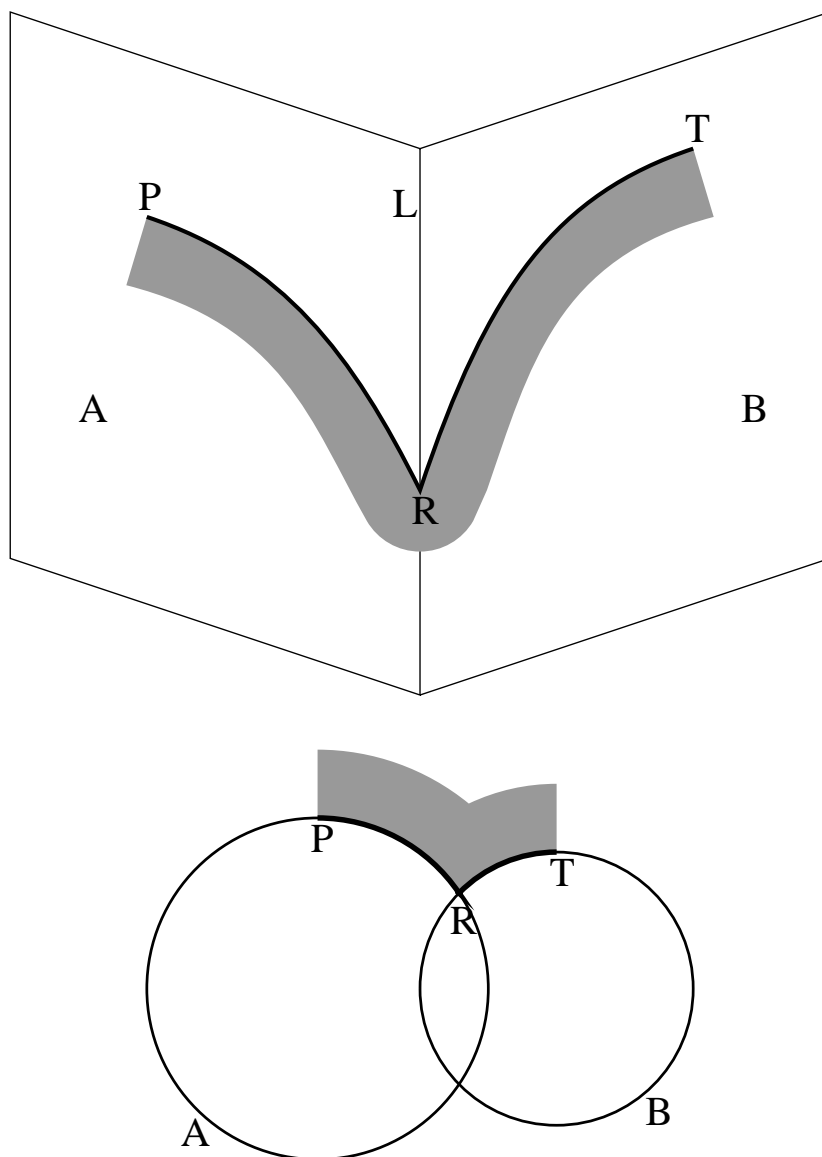


**Figure 5.** Case C1: $\pi + \gamma_A - \gamma_B < \alpha < \pi + \gamma_A + \gamma_B$. Upper, surface trace. The trace

meets the corner obliquely. Lower, Gauss map.

Subcase C1. (see figure 5) On the upper $w$ half-plane, define

$$g(w) = g_R + ie^{i\theta}w^{2k+1-\theta/\pi}.$$

Then by Lemma 1, the trace condition on the real $w$ axis is

$$\Im\left[(g(w) - g_R)^2 f(w)/(dg/dw)\right] = 0,$$

or

$$\Im\left[-e^{2i\theta}w^{4k+2-2\theta/\pi}f(w)(-i)e^{-i\theta}\frac{\pi}{(2k+1)\pi - \theta}w^{-2k+\theta/\pi}\right] = 0.$$

After dropping real factors, this boils down to

$$\Re\left[e^{i\theta}w^{-\theta/\pi}f(w)\right] = 0.$$

Thus

$$f(w) = -ie^{-i\theta}w^{\theta/\pi}h(w).$$

For smoothness, take $\zeta = w^{1+\theta/\pi}$. Then

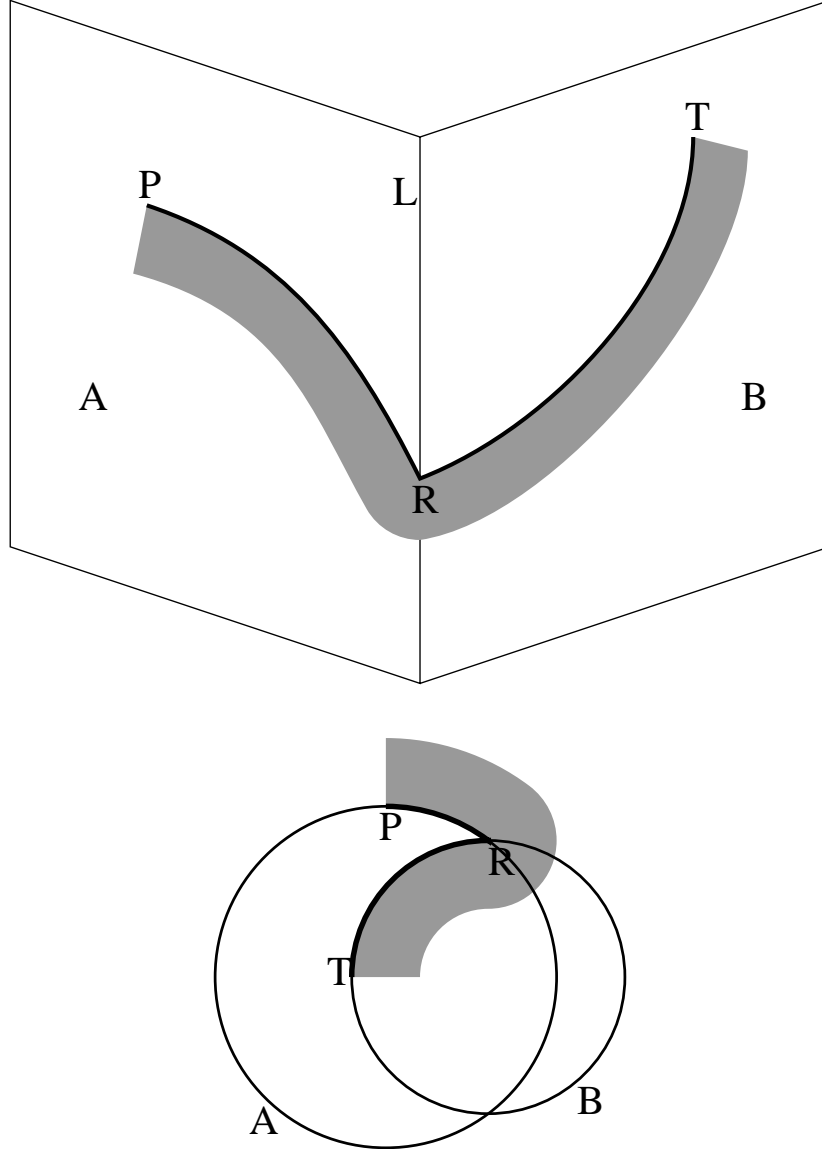$$g(w) - g_R \propto \zeta^{\frac{2k+1-\theta/\pi}{1+\theta/\pi}}.$$

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**Figure 6.** Case C2: $\pi + \gamma_A - \gamma_B < \alpha < \pi + \gamma_A + \gamma_B$. Upper: surface trace. Lower: Gauss map. There is a similar subcase with convexity reversed on the half-planes and with the Gauss map wrapping around the left of R.

Subcase C2. (see figure 6) This works out similarly, with

$$g(w) = g_R - ie^{i\theta}w^{2k+2-\theta/\pi},$$
$$f(w) = -ie^{-i\theta}w^{\theta/\pi}h(w),$$
$$\zeta = w^{1+\theta/\pi},$$
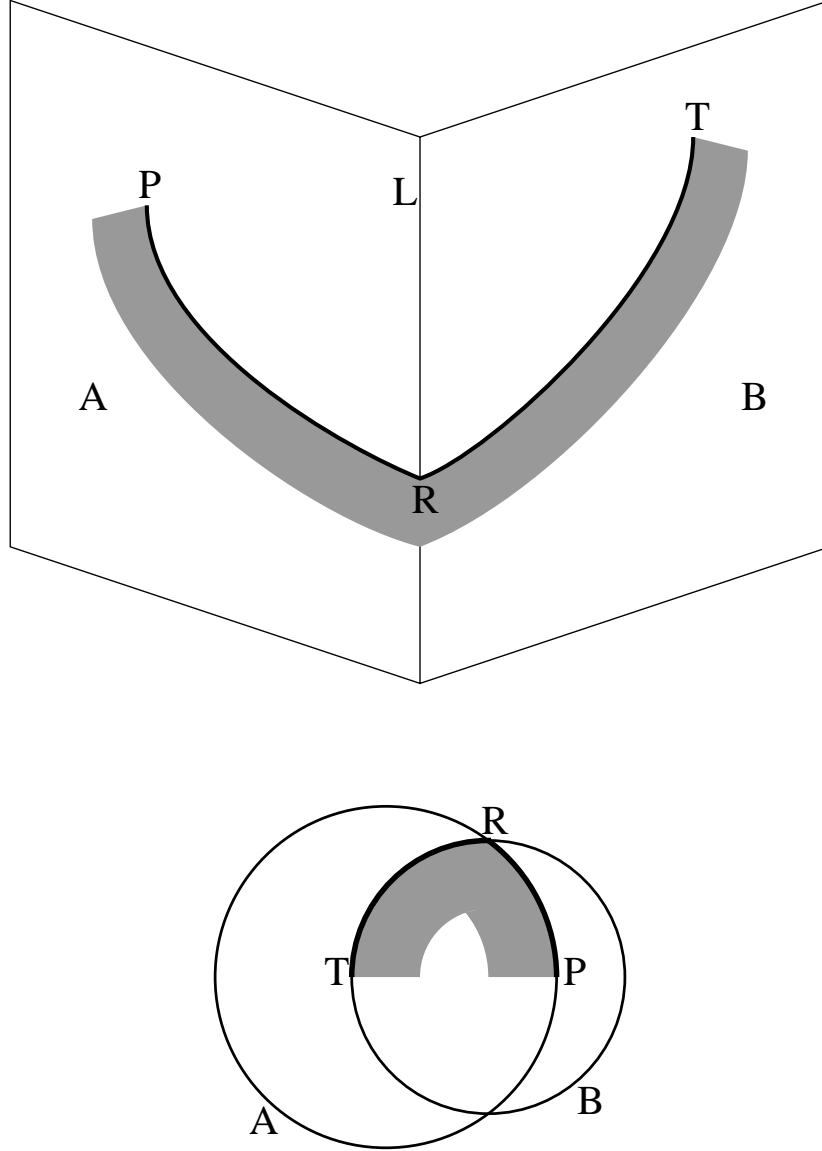$$g(w) - g_R \propto \zeta^{\frac{2k+2-\theta/\pi}{1+\theta/\pi}}.$$

**Figure 7.** Case C3: $\pi + \gamma_A - \gamma_B < \alpha < \pi + \gamma_A + \gamma_B$. Upper, surface trace. The trace meets the corner obliquely. Lower, Gauss map.

Subcase C3. (see figure 7) This works out similarly again, with

$$g(w) = g_R - ie^{i\theta}w^{2k+1-\theta/\pi},$$
$$f(w) = -ie^{-i\theta}w^{\theta/\pi}h(w),$$
$$\zeta = w^{1+\theta/\pi},$$
$$g(w) - g_R \propto \zeta^{\frac{2k+1-\theta/\pi}{1+\theta/\pi}}.$$

## 6. Case D: $\alpha = \pi + \gamma_A - \gamma_B$.

Here the Gauss circle of $B$ is internally tangent to that of $A$. The trace on the wall crosses $L$ tangentially at the point $R$. The Gauss image wraps all the way around $R$, as shown in figure 8.
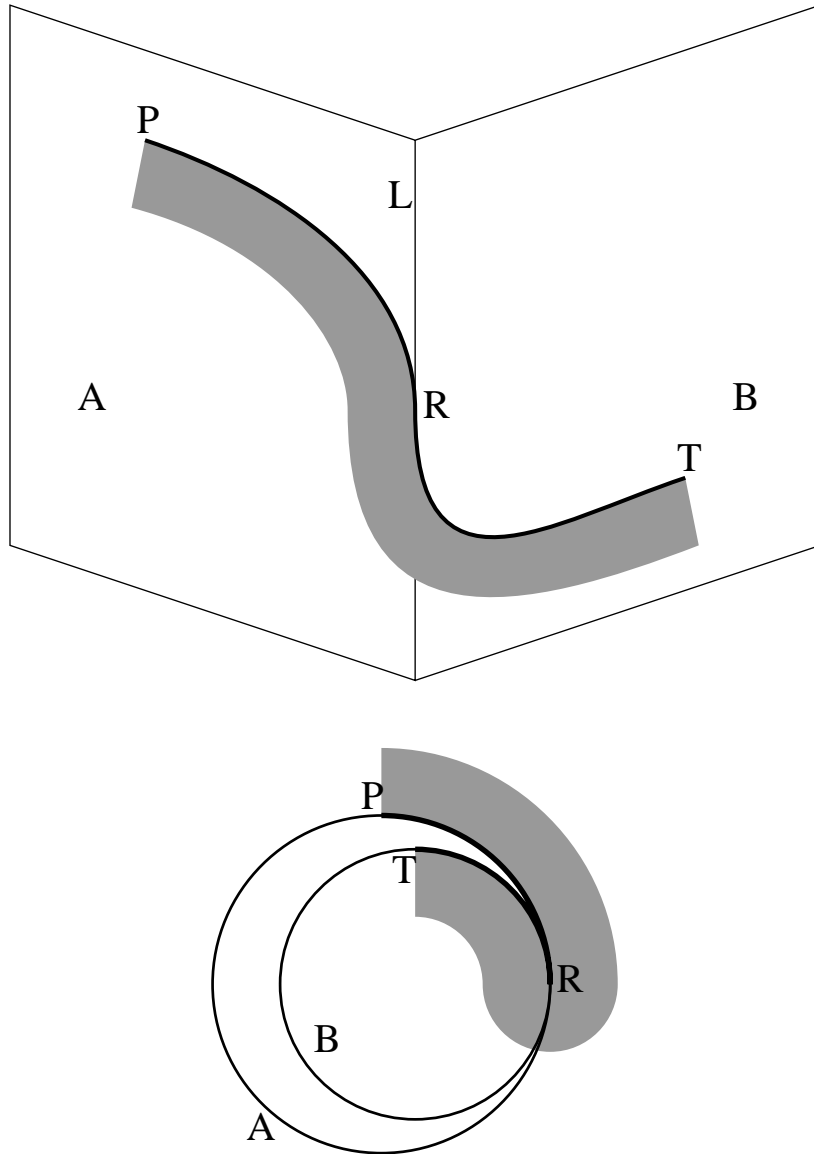


**Figure 8.** Case D: $\alpha = \pi + \gamma_A - \gamma_B$. Upper, surface trace. The trace crosses the corner at a point of tangency. Lower, Gauss map.

To study the Weierstrass representation at the critical point $R$, orient the corner so that $L$ is the $y$-axis and the common normal (at $R$) points vertically upward and the Gauss circles are in the positive $x$ half-space. The Gauss circles pass through the north pole of the Gauss sphere and the image in the $g$ plane wraps down the right side of the line $\Re g = \cot \gamma_A$ to $-i\infty$, all the way around $\infty$ (perhaps multiple times), and back up the

left side of the line $\Re g = \cot \gamma_B$. In the upper $w$ half-plane, define the Gauss map to be

$$g(w) = -iw^{-2-2k} + \frac{\cot \gamma_B - \cot \gamma_A}{i\pi} \log w + \cot \gamma_A.$$

This wraps the Gauss image an extra $k$ times around the north pole with the positive $w$ axis mapping to the $A$ trace and the negative $w$ axis mapping to the $B$ trace. As in case B, the trace conditions for $f$ on the real $w$ axis become $\Re f(w) = 0$. The surface at $R$ subtends an angle of $\pi$, so $g(w)^2 f(w) dw \propto dw$, which means $f(w)$ has a zero of order $4k + 4$. Hence the general Weierstrass representation is

$$g(w) = -iw^{-2-2k} + \frac{\cot \gamma_B - \cot \gamma_A}{i\pi} \log w + \cot \gamma_A,$$
$$f(w) = iw^{4k+4} h(w).$$

For smoothness, let $\zeta = w$. Again as in case B, look at the smoothness of $1/g$. Then, essentially,

$$1/g \approx \zeta^{2+2k} [1 + a\zeta^{2+2k} \log \zeta + b\zeta^{2+2k}]^{-1}.$$

Hence

$$1/g \approx \zeta^{2+2k} - a\zeta^{4+4k} \log \zeta - b\zeta^{4+4k} + \dots.$$

Thus the normal is almost of class $C^{4+4k}$.

## 7. Case E: $\alpha > \pi - \gamma_A + \gamma_B$.

Here the Gauss trace can go along the stable part of the Gauss circle of $L$ from the outside of one Gauss circle to the inside of the other. Hence the trace itself has to go along line $L$ to get from one half-plane to another. Referring to figure 9, along the segment QS the orientation of the tangent plane rotates from proper angle for $A$ to the proper angle for $B$.
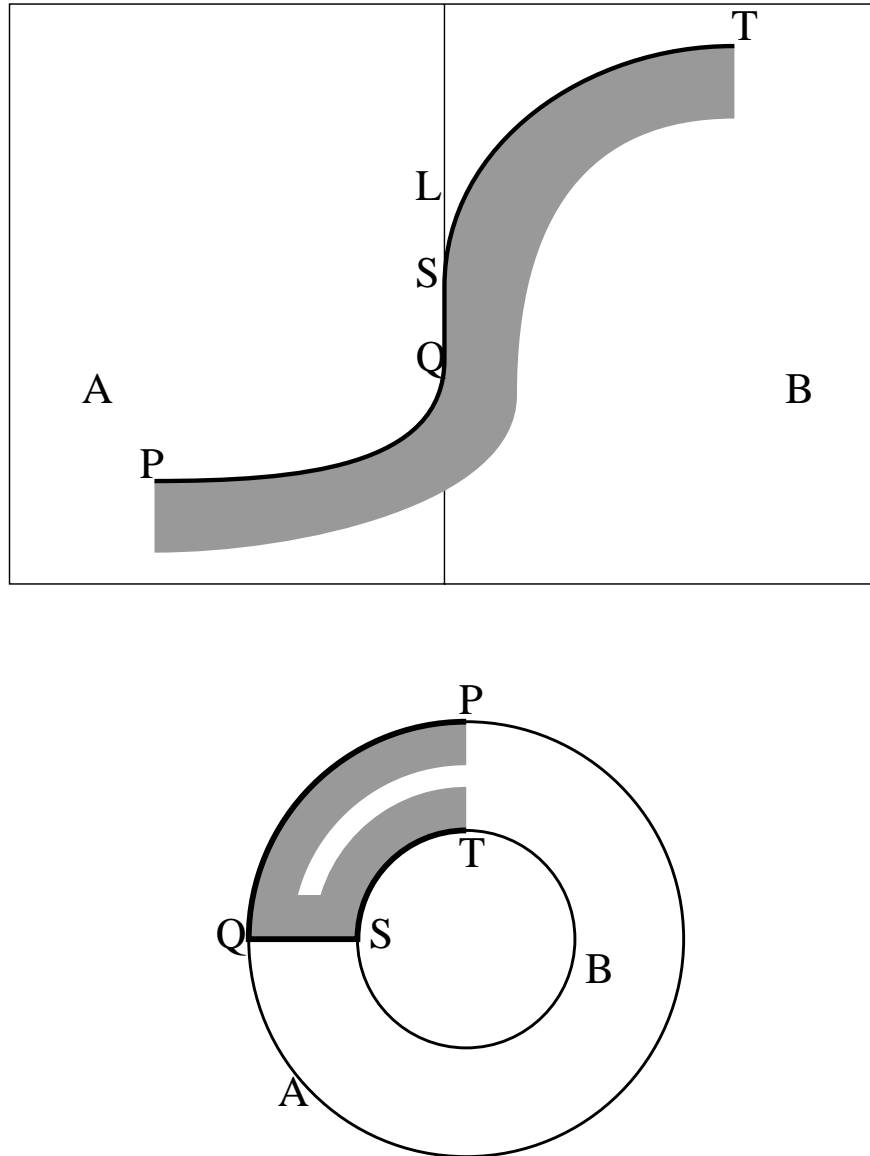




**Figure 9.** Case E: $\alpha > \pi - \gamma_A + \gamma_B$. Upper, surface trace. The trace runs along the corner while its normal turns. Only $\alpha = \pi$ is pictured here, but this case covers a wide range of $\alpha$. It includes all concave and some convex corners.

The general form of the Weierstrass representation around $Q$ and $S$ is the same as in case A since the local geometry there only involves one half-plane.

This is the most convenient case in which to study the actual length of the trace along the corner. An informative exact solution can be found easily for the case $\alpha = \pi$. We will take the non-wall boundary of the surface to be a straight wire parallel to the plane at an angle $\theta$ to the line $L$, $\theta$ being positive clockwise, and at some distance $H$ from the plane. Note that for $\theta < 0$ the surface is the mirror image in the $x, z$-plane of the surface for $|\theta|$. To get a nice domain, orient the coordinate axes so that positive $x$ is to the left and positive $y$ is upward. This means we are looking up through the south pole of the Gauss sphere, and on the $g$ plane the domain is in an annulus. The radii of the circles can be calculated from the values of $g$ at points $Q$ and $S$:

$$\frac{g^2 - 1}{g^2 + 1} = -\cos\gamma,$$

so

$$g_Q = \tan(\gamma_A/2), \qquad g_S = \tan(\gamma_B/2).$$

For the parameter, take $w = \log g$. Then the domain in the $w$-plane is a rectangle. The trace condition for $f(w)$ may be taken to be $dz = 0$ or $\Re[2g(w)f(w)dw] = 0$ on the whole trace. Since $g = e^w$, this means that on the top and bottom sides of the rectangle (where $dw$ is real) we have $\Re[e^w f(w)] = 0$, and on the left and right sides (where $dw$ is imaginary) we have $\Im[e^w f(w)] = 0$. Hence $e^w f(w)$ must have zeroes or poles at the corners. We want poles at $T$ and $P$ since the surface goes off to infinity there. We want zeroes at $Q$ and $S$. By the reflection principle, $e^w f(w)$ must be doubly periodic. All this points to elliptic functions; in particular,

$$e^w f(w) = C \operatorname{sn}\left(2K\frac{w - w_S}{w_Q - w_S}\right),$$

where $C$ is an arbitrary real constant which sets the scale of the surface and the condition on the quarter periods $K$ and $iK'$ is

$$K' = 2K\frac{\theta}{w_Q - w_S}.$$

(See chapter 16 of [AS] for information on sn.) Hence

$$f(w) = Ce^{-w} \operatorname{sn}\left(2K\frac{w - w_S}{w_Q - w_S}\right).$$

Figure 10 shows the ratio $H/D$ between the distance $H$ of the wire from the wall and the length $D$ of the trace along $L$ for $\gamma_A = \pi/2$ and various values of $\gamma_B$ and $\theta$. Note that extremely large values are reached for very run-of-the-mill angles. The maximum value of $\theta$ shown is $\pi$, but there is no reason to stop there if you're willing to move the boundary line between $A$ and $B$ to follow the trace where it wants to go. In the limit of an infinitely high wire ($H, \theta \to \infty$), one gets

$$f(w) = Ce^{-w} \sin\left(\pi\frac{w - w_S}{w_Q - w_S}\right),$$

20

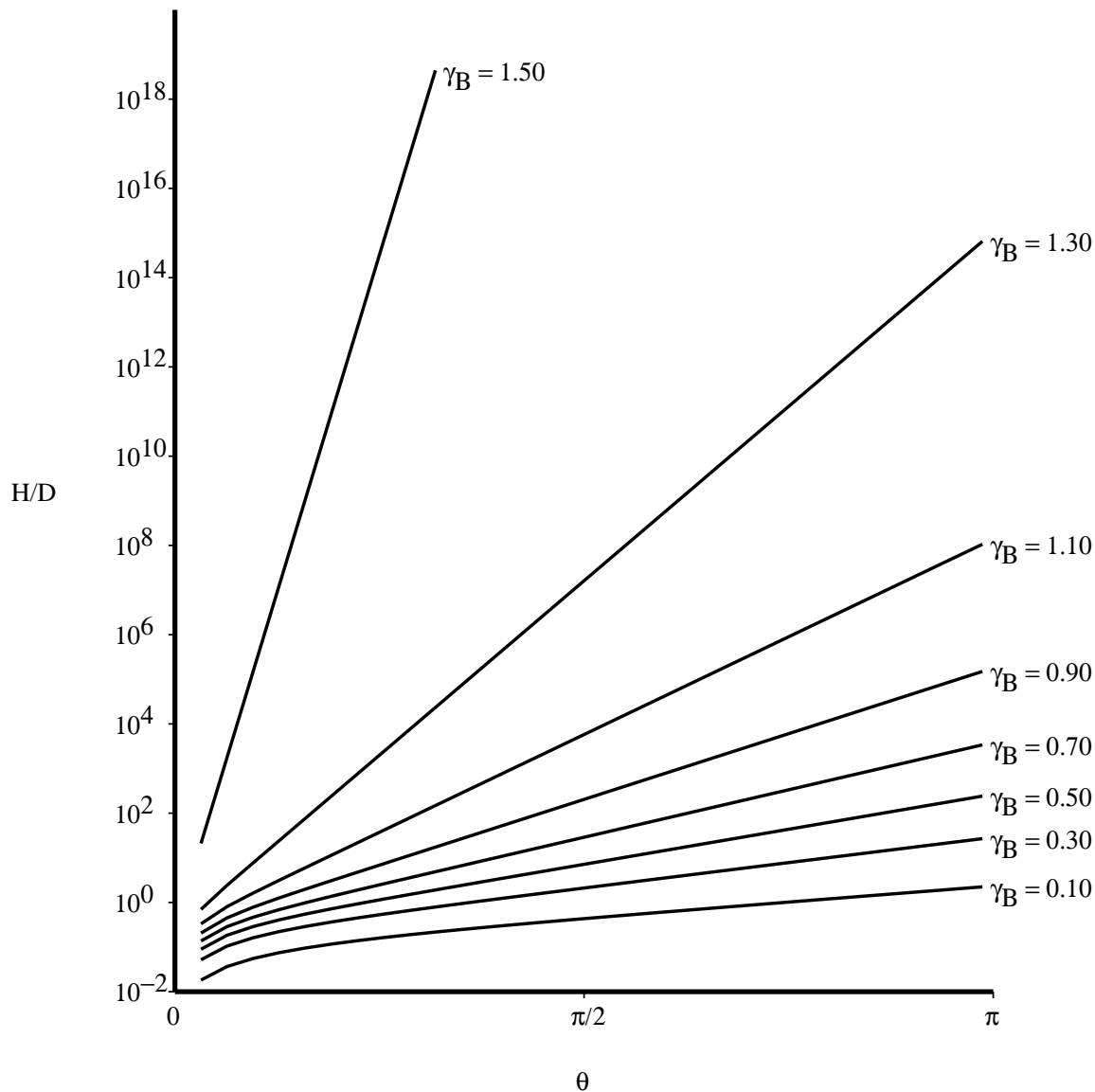and the trace blooms out into an exponential spiral.



**Figure 10.** Case E exact solution. Ratio of wire height $H$ to corner trace length $D$ for $\gamma_A = \pi/2$.

The case $\gamma_A = \pi/2$ is of special interest because by adjoining the mirror image surface on the other side of the wall, one gets a surface that spans a corner of angle $2\pi$ with equal contact angles $\gamma_B$ on both sides of the half-plane. For $\gamma_B = \pi/3$, this is just the case that crops up in the examples mentioned above of free ends of zero thickness boundary wire.

This case illustrates the other of the possible behaviors that Lancaster [L] derived for radial limits at a non-convex corner for a discontinuous trace. In figure 9, let the $x, y$ plane be endowed with polar coordinates $r, \theta$ with origin at the corner, plane $A$ at $\theta = 0$, and plane $B$ at $\theta = \alpha$, and let the surface be represented as $z = f(r, \theta)$. The behavior is that

the radial limit is constant on an interval of $\theta$, then monotone then finally constant. Here

$$Rf(\theta) = \begin{cases} z_Q, & \text{if } 0 \le \theta \le \pi - \gamma_A, \\ \text{increasing}, & \text{if } \pi - \gamma_A < \theta < \alpha - \gamma_B, \\ z_S & \text{if } \alpha - \gamma_B \le \theta \le \alpha. \end{cases}$$

## 8. Case F: $\alpha = \pi - \gamma_A + \gamma_B$.

Here we have a convex corner with the Gauss circle of $B$ internally tangent to that of $A$. The trace crosses $L$ tangentially. See figure 11. Exact solutions work out much as in case B. In fact, case B solutions can be made by pasting two of these solutions together along the midline of case B.
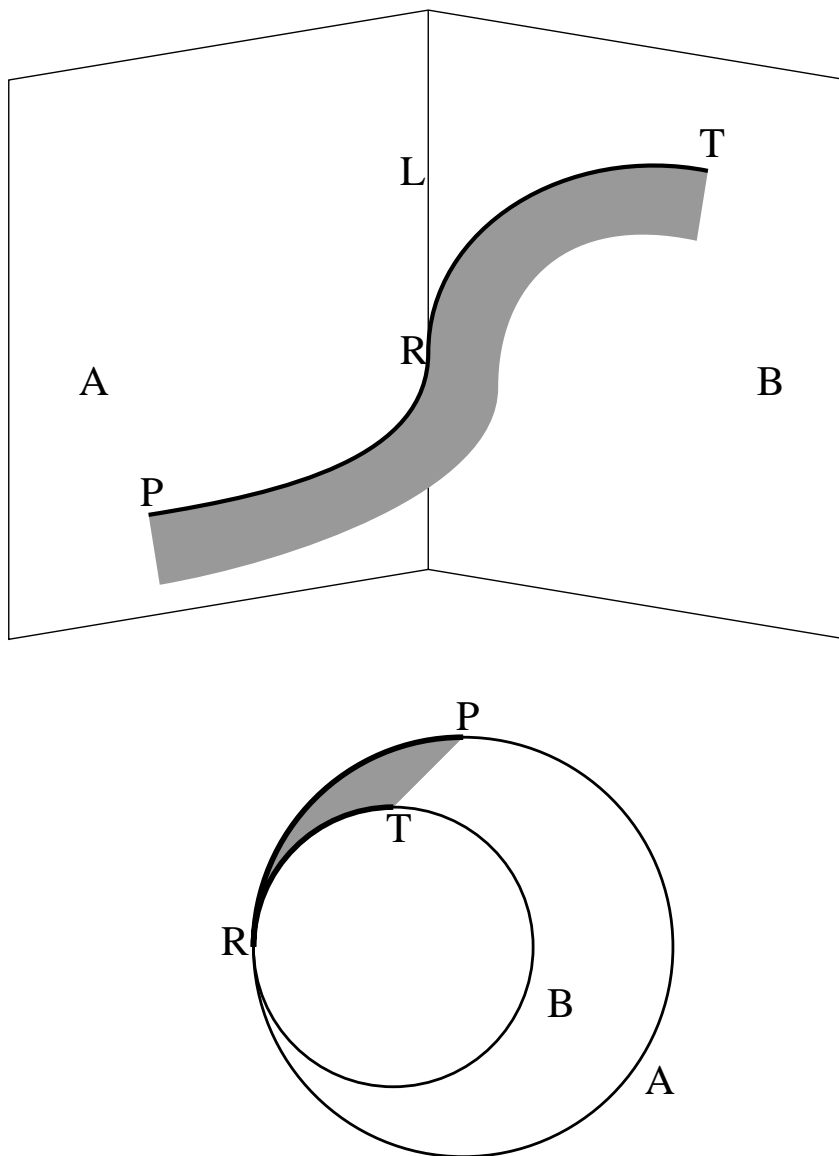


**Figure 11.** Case F: $\alpha = \pi - \gamma_A + \gamma_B$. Upper, surface trace. Lower, Gauss map. The trace crosses the corner at a point of tangency R.

To study the Weierstrass representation at the critical point $R$, orient the corner so that $L$ is the $y$-axis and the common normal (at $R$) points vertically upward and the Gauss circles are in the positive $x$ half-space. Then the Gauss circles pass through the north pole of the Gauss sphere and the image in the $g$ plane is the strip $\cot \gamma_A \leq \Re g \leq \cot \gamma_B$ with $g_R = +i\infty$, with an arbitrary number of wraps around the north pole. In the upper $w$ half-plane, define the Gauss map to be

$$g(w) = iw^{-2k} + \frac{\cot \gamma_B - \cot \gamma_A}{i\pi} \log w + \cot \gamma_A.$$

This wraps the Gauss image $k$ times around the north pole with the positive $w$ axis mapping to the $A$ trace and the negative $w$ axis mapping to the $B$ trace. As in case B, the trace conditions for $f(w)$ on the real $w$ axis become $\Re f(w) = 0$. The surface at $R$ subtends an angle of $\pi$, so $g(w)^2 f(w) dw \propto dw$, which means $f(w)$ has a zero of order $4k$. Hence the general Weierstrass representation is

$$g(w) = iw^{-2k} + \frac{\cot \gamma_B - \cot \gamma_A}{i\pi} \log w + \cot \gamma_A,$$
$$f(w) = iw^{4k} h(w).$$

For smoothness, let $\zeta = w$. Again as in case B, look at the smoothness of $1/g$. Then essentially,

$$1/g \approx \zeta^{2k}[1 + a\zeta^{2k} \log \zeta + b\zeta^{2k}]^{-1}.$$

Hence

$$1/g \approx \begin{cases} 1/(1 + a \log \zeta + b), & \text{if } k = 0; \\ \approx \zeta^{2k} - a\zeta^{4k} \log \zeta - b\zeta^{4k} + \dots, & \text{if } k > 0. \end{cases}$$

Thus the normal is of class $C^0$ or almost of class $C'^{4k}$ respectively.

As in case B, sample solutions are easy to find in the strip in the $g$ plane. One function that works is

$$f(g) = \sec\left(\pi \frac{g - \cot \gamma_A}{\cot \gamma_B - \cot \gamma_A}\right).$$

The shape of the trace is approximately

$$|x| \propto |z| \propto \left| \frac{y}{|\log y|} \right|.$$

**9. Case G:** $\pi - \gamma_A - \gamma_B < \alpha < \pi - \gamma_A + \gamma_B$.

This is the convex corner counterpart to case C. Once again, there is a common normal to both sides, and the trace reaches the corner obliquely. There are several subcases, as shown in figures 12, 13, and 14, depending on the concavity of the trace.

The general form of the Weierstrass representation is like that in case C. Here, though, the angle subtended by the surface at $R$ is $\pi - \theta$.
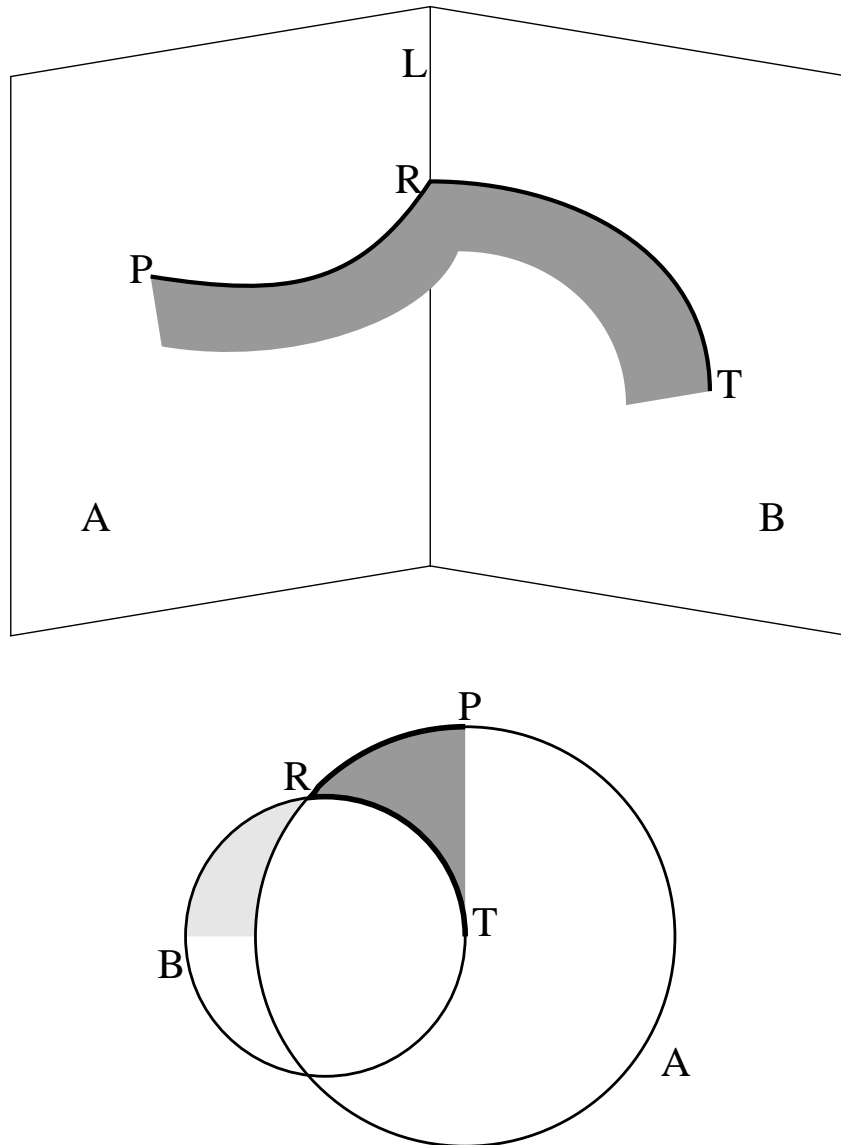


**Figure 12.** Case G1: $\pi - \gamma_A - \gamma_B < \alpha < \pi - \gamma_A + \gamma_B$. Upper, surface trace. The trace meets the corner obliquely. Lower, Gauss map. There is also a case where the domain is on the opposite side of R, lightly shaded.

Subcase G1. (see figure 12) This works out with

$$g(w) = g_R + iw^{2k+\theta/\pi},$$
$$f(w) = -iw^{-\theta/\pi}h(w),$$
$$\zeta = w^{1-\theta/\pi},$$
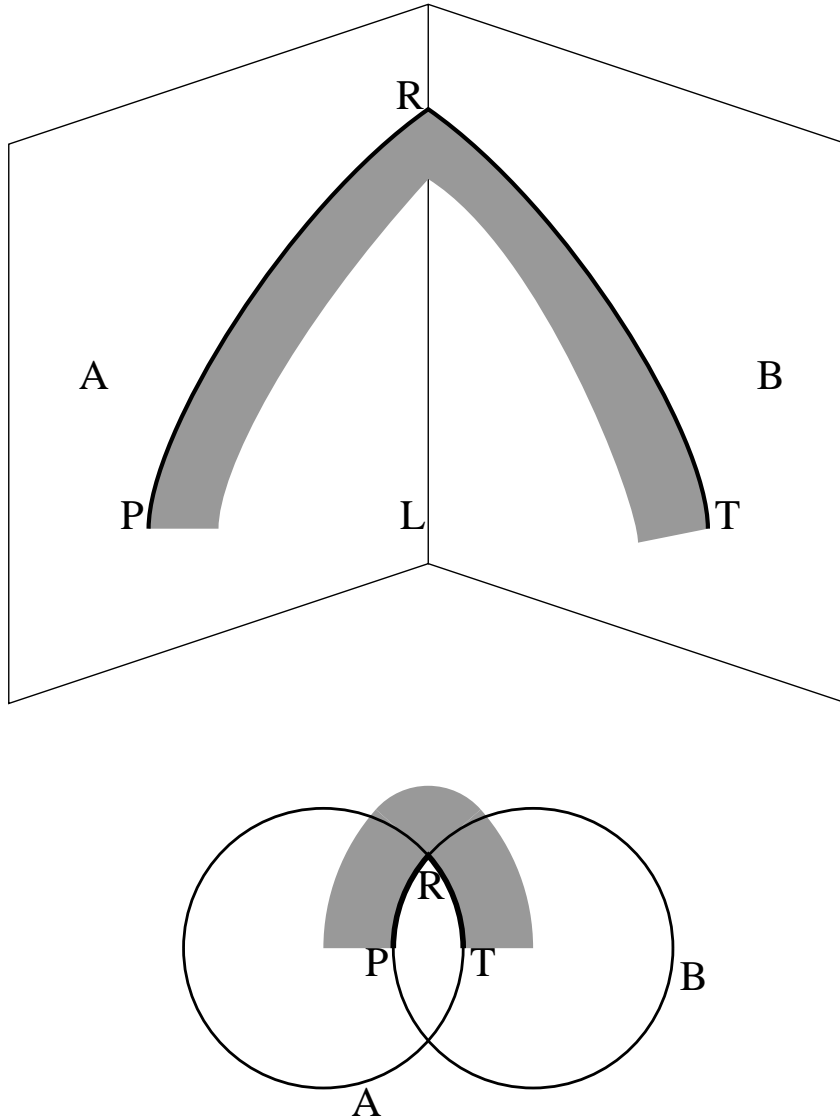$$g(w) - g_R \propto \zeta^{\frac{2k+\theta/\pi}{1-\theta/\pi}}.$$



**Figure 13.** Case G2: $\pi - \gamma_A - \gamma_B < \alpha < \pi - \gamma_A + \gamma_B$. Upper, surface trace. The trace meets the corner obliquely. Lower, Gauss map.

Subcase G2. (see figure 13) This works out similarly, with

$$g(w) = g_R - iw^{2k+1+\theta/\pi},$$
$$f(w) = -iw^{-\theta/\pi}h(w),$$
$$\zeta = w^{1-\theta/\pi},$$
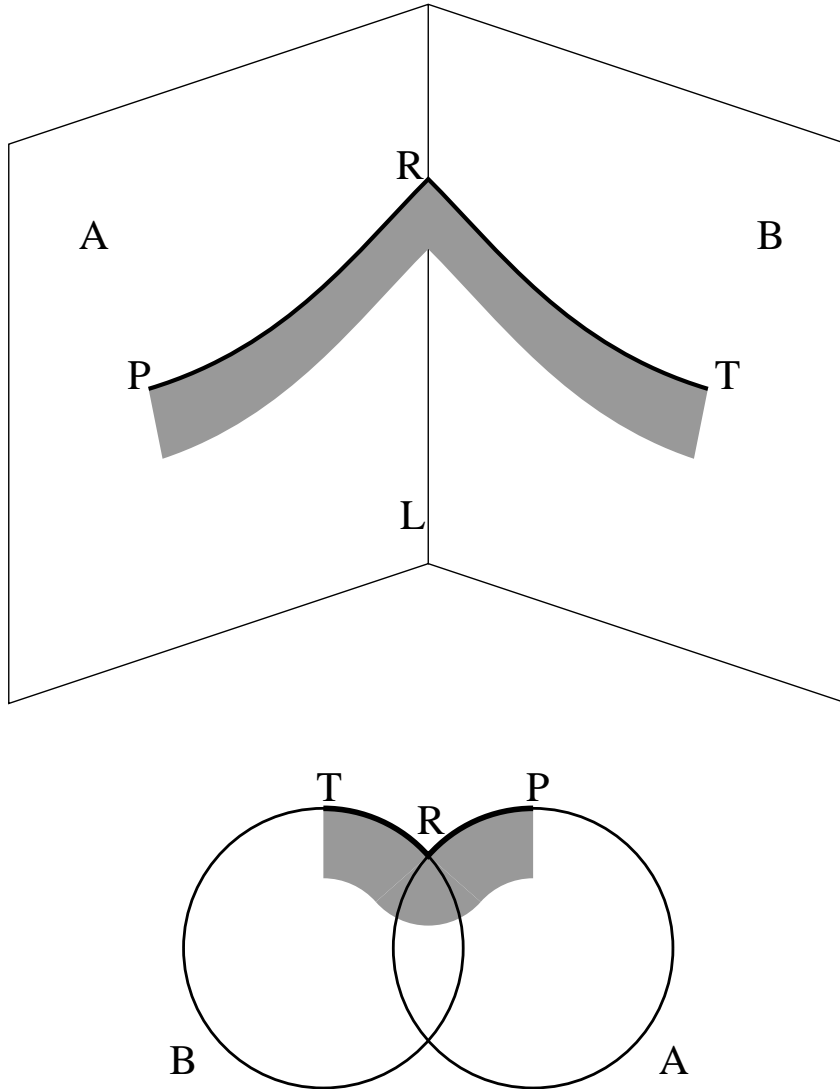$$g(w) - g_R \propto \zeta^{\frac{2k+1+\theta/\pi}{1-\theta/\pi}}.$$



**Figure 14.** Case G3: $\pi - \gamma_A - \gamma_B < \alpha < \pi - \gamma_A + \gamma_B$. Upper, surface trace. The trace meets the corner obliquely. Lower, Gauss map.

Subcase G3. (see figure 14) This works out similarly again, with

$$g(w) = g_R - iw^{2k+1+\theta/\pi},$$
$$f(w) = -iw^{-\theta/\pi}h(w),$$
$$\zeta = w^{1-\theta/\pi},$$
$$g(w) - g_R \propto \zeta^{\frac{2k+1+\theta/\pi}{1-\theta/\pi}}.$$

## 10. Case H: $\alpha = \pi - \gamma_A - \gamma_B$.

In this case, there is a single common surface normal $\vec{N}$ for the two planes, and it is perpendicular to the corner line $L$. The geometry is such that if $H$ is a plane perpendicular to $\vec{N}$ inside the corner, then the strip of $H$ cut off by the corner is a minimal surface with the proper contact angles on the half-planes.

It turns out that the trace cannot cross the corner. If one tries to construct a Weierstrass representation that does, one runs into a contradiction. Suppose a surface existed with its trace crossing $L$. The geometry of the situation forces the surface to form a cusp at a point $R$ on $L$, at which the surface subtends a zero angle. The Gauss map analysis is like that for case D, leading to $f(w)$ real on the real $w$ axis and $g(w) \approx w^{-2}$. But then there is no way to choose the leading power of $f(w)$ so that the surface subtends a zero angle.

Another way to see that such a surface is impossible is to use the strip of $H$ mentioned above to cut off the piece of the surface next to the corner. Move the piece upward a bit, filling in the gap created with a piece of the plane $H$. The energy of this surface is exactly the same as the original since the energy saved moving the trace up the wall is the same as the area of the gap. By the analyticity of minimal surfaces, this surface is not a minimal surface since it has a plane piece but does not lie entirely in that plane. Therefore a smaller surface exists nearby and the original surface was not minimal.

One possible behavior is for the surface to asymptotically approach a plane strip running up the corner. This differs from capillary surfaces, in which gravity keeps the surface height bounded, and the trace does cross the corner with a continuous surface normal [TL].

An interesting fact is that the non-wall boundary of a bounded minimal surface lies entirely inside $H$, then so must the surface. If some part of the surface lies outside of $H$, then the variation vectorfield that projects the outside of $H$ opposite $\vec{N}$ to the wall and $H$ shows that the surface is not minimal. The corresponding fact is not true if the non-wall boundary lies outside $H$, as may be seen by considering a slice of a catenoid bounded by a circle parallel to one half-plane with its trace on that half-plane also being a circle.

## 11. Case I: $\alpha < \pi - \gamma_A - \gamma_B$.

A similar argument to case H shows that the trace cannot cross the corner. But here there isn't even a solution that is a strip of plane running up the corner.

**12. A degenerate case: $\alpha = \pi$ and $\gamma_A + \gamma_B = \pi$.**

This case describes a surface wrapping around the edge of a half-plane for which the Gauss circles of the sides coincide. There are two general possibilities. First, this can be just an ordinary example of case E with the surface normal rotating as the trace runs along the edge of the half-plane. Second, the Gauss trace can jump from one circle to the other at a continuum of possible points. This can be viewed as an example of case B with at least one extra turn of the Gauss image around $g_R$, the lowest order case discussed in detail in case B above being non-existent when the Gauss circles coincide.

This case with $\gamma_A = \gamma_B = \pi$ corresponds to the situation treated in [HN]: a soapfilm wrapping around the edge of a half-plane. Much of the analysis there also applies to this slightly more general case. In particular, the even order branch points of [HN] correspond to my case E and the odd to my case B.

**13. Contact angles 0 and $\pi$.**

If the contact angle on a half-plane is 0 or $\pi$, then one fluid will completely wet the half-plane and there will be no trace on that half-plane. However, there can still be a trace along the corner and on the other half-plane. One could also have contact angle 0 on both planes and still have a trace on the corner if $\alpha > \pi$. There can be points where the trace on $L$ reverses direction, to which the analysis of point $R$ in case A applies.

**14. A wire meeting a plane.**

Suppose a boundary wire of a surface meets a plane at an angle $\theta$, measured between the plane and the wire. Let $\gamma$ be the contact angle on the plane. If $\theta \leq \gamma$, then the wire and plane have a common normal, and the surface can have a nice tangent plane at the intersection. If $\theta > \gamma$, then there is no such nice possibility. The mathematical solution here is that the trace on the plane makes an inward spiral infinitely many times around the wire. For a sample exact solution, let the plane be horizontal and the wire vertical. Take the domain to be $0 \leq \Re w \leq \cot(\gamma/2)$. Then one can take

$$g(w) = e^w, \qquad f(w) = i \exp\left(-w + \frac{i\pi w}{2\cot(\gamma/2)}\right).$$

The trace makes an exponential spiral whose radius decrease by a factor of $\pi^2/\cot(\gamma/2)$ each revolution.

If the surface is the boundary between two fluids, then an infinite spiral is not possible. The spiral would run into the fluid boundary on the other side of the wire instead. The net result is that the surface does not reach the intersection of the wire and plane. The surface departs from the wire before reaching the plane, much like the surface departs the wire where a free end goes through the surface. With the same configuration as in the previous paragraph, an exact solution is

$$g(w) = e^w, \qquad f(w) = e^{-w} \sin\left(\frac{\pi w}{2\cot(\gamma/2)}\right).$$

## 15. Conclusion.

The behavior of minimal surfaces at a corner should also be the behavior of capillary surfaces to a large degree. The Gauss trace considerations remain the same, and the extra forces are negligible at a small scale. Perturbations such as curved walls, volume constraints, or a continuously varying contact angle on each half-plane also should not affect the small-scale behavior significantly. One of the themes of this paper is that when there is not a common surface normal at a corner, the trace must run along the corner while the normal turns, and that should apply to capillary surfaces also.

It is interesting to note that in the original case of a wire going through a soap film that there are no unusual types of tangent cones on the wire. At each point on the wire, the tangent cone of the surface is either a half-plane or two half-planes at angles of $2\pi/3$ or greater.

How does the infinitesimally thin wire solution match up with the thick wire solutions as the wire diameter goes to zero? From afar, the very thin wire solution seems to have the triple line meet the wire tangentially, but magnification shows that the triple line makes a tiny sharp curve to actually meet the wire perpendicularly. Likewise, from afar it looks like the skirt leaves the wire tangentially, but magnification shows a sharp curve near the wire so the skirt meets the wire perpendicularly.

The type of analysis done in this paper also illuminates the behavior of surfaces where several boundary wires meet. For example, if wires meet in a $\perp$-shape, then the surface will look much like the free-end surface above the point where the surface curls away from the free end. The triple line will come in tangent to the vertical stem of the $\perp$ some short distance away from the intersection point, and the angle between the surfaces will grow from $2\pi/3$ to $\pi$ between there and the intersection point.

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