SOAP FILMS AND COVERING SPACES

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ABSTRACT. A new mathematical model of soap films is proposed, called the "covering space model." The two sides of a film are modelled as currents on different sheets of a covering space branching along the film boundary. Hence a film may be seen as the minimal cut separating one sheet of the covering space from the others. The film is thus the oriented boundary of one sheet, which represents the exterior of the film. As oriented boundaries, films may be calibrated with differential forms on the covering space, a version of the min-cut, max-flow duality of network theory. This model applies to unoriented films, films with singularities, films touching only part of a knotted curve, films that deformation retract to their boundaries, and other examples that have proved troublesome for previous soap film models.

1. INTRODUCTION

§1.1. Soap films.   Soap films are one of the major wonders of the universe. In their thinness, they approach the geometrical ideal of a two-dimensional surface as few geometrical ideals are ever approached in the real world. They have attracted the attention of mathematicians for centuries (see [NJ] for this glorious history), yet they are still full of surprises, such as the films that can form on a simple trefoil knot. Many mathematical models of soap films have been proposed, but so far none has been able to cope with all the observed phenomena. This paper proposes a new model that is much closer to the physical structure of soap films and so is able to cope with more of the variety seen in real soap films.

The focus of this paper is on soap films rather than closely related problems such as soap bubbles and capillary surfaces, although the model proposed here does extend to these. The model is of mathematically ideal soap films, disregarding film thickness, film drainage, and Plateau borders. However, the inspiration for the new model does come from the physical fact that soap films are really double films.

The key property of a soap film is that it minimizes area while spanning a boundary. Various notions of "area" and "spanning" give rise to various mathematical models of soap films: manifolds, rectifiable sets, integral currents, and varifolds, to

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name a few. The fundamental problems to be addressed are first, the representation of films; second, the existence of minimal films; third, a method of construction; fourth, a way to prove that a given film is indeed the minimum; and fifth, the structure of films in terms of smoothness and the nature of singularities.

§**1.2. Summary.**   This paper proposes to model a soap film as a certain type of current on a covering space of the complement of the film boundary. Physically, a soap film is really made of two oppositely oriented surfaces with a thin layer of liquid between. Each surface is orientable, and therefore current-like, even where there are triple junctions in the film. It is natural to model the film with one current for each side. The modelling problem is to get the currents for both sides to be in the same place without their opposite orientations cancelling each other out. The new model does this by having the currents for the different sides live on different sheets of a covering space. It will be seen that the soap film may be visualized as the minimal cut separating one sheet from the others. The sides of the film are the oriented boundary of that one sheet, which represents the exterior of the film. Since the film is an oriented boundary, calibration methods (essentially the minimum cut, maximum flow duality of standard network theory) are available to prove that even unorientable soap films with singularities are absolutely area minimizing in some sense.

Section 2 gives some examples of soap film phenomena and past mathematical models. Section 3 gives some background on forms and currents. Section 4 motivates the covering space model by discussing a simple example, the triple junction. Section 5 describes the covering spaces involved. Section 6 formulates various soap film problems and uses compactness to show the existence of solutions. Section 7 shows how the new model applies to examples, including cases that have given difficulties to previous models. Section 8 defines calibrations, shows how they are used to prove minimality, and proves their existence in the appropriate problems. Section 9 shows how calibrations can be used to prove uniqueness. Section 10 is a brief look at regularity, showing that films correspond often to $(\mathbf{M}, 0, \delta)$-minimal sets and always to minimal varifolds. Section 11 looks at what one can say about films on particular boundaries (barrier theorems, for example), and gives a new proof that knotted curves support films that do not touch the entire curve. Section 12 discusses ways of designing covering spaces to acheive desired effects, such as constructing a covering space for a given film. Section 13 discusses the relation between the covering space model and some other soap film models. The concluding section summarizes some open questions and directions for future research.

The persistent theme running through this paper is duality: dual linear spaces, dual norms, and the duality of maximization and minimization problems. The existence and proof-of-minimality problems can often be solved using this duality. Upper bounds on area can be proved by exhibiting actual surfaces. Lower bounds can be found by considering the dual maximization problem and exhibiting feasible solutions thereto. The dual problem for minimizing the mass of a current spanning a boundary is maximizing the flux of a divergenceless vectorfield through the boundary with the constraint that the vectorfield has maximum magnitude 1. This is a continuous version of the network theory max flow - min cut theorem. If one can exhibit dual solutions with the same value of the objective function, then

one has an absolutely minimizing surface. The fact that there is a dual maximization problem in the sense of linear programming is one of the main benefits of the representation of soap films as currents. Many of the results of this paper were discovered through contemplation of standard linear programming techniques applied to discretized versions of the problems. Unfortunately, linear programming yields real-valued solutions instead of integer-valued, so the currents involved must be real currents, a much vaster class than integral currents. But the integral current minimizer often happens to also be the real current minimizer. If not, finding the integral current solution can be difficult, as integer programming is in general much more difficult than linear programming.

The author has written a computer program, named `Polycut`, that visualizes three dimensional covering spaces. It currently runs only on Silicon Graphics workstations. It is available by anonymous ftp from `geom.umn.edu` as the file `pub/polycut.tar.Z`. Most of the covering space examples in this paper are included. `Polycut` shows what the covering space would look like from the inside, assuming the different sheets have different colors at infinity. It can superimpose the corresponding soap film to show how the film works as a cut.

§1.3. Acknowledgements. This model is an outgrowth of the paired calibration techniques discovered by Lawlor and Morgan [LM] and used in [B1]. I wish to thank Fred Almgren for getting me started thinking about calibration techniques, and Frank Morgan for clarifying discussions and challenging examples.

2. Phenomena and Models

§2.1. Phenomena.
A vast diversity of soap film phenomena may be seen by dipping wire frames into soap solution [AT],[BC],[IC]. Some examples are shown in figure 1. A simple wire loop often gives a film that is topologically a disk (figure 1A). But if the loop is twisted into the boundary of a Mobius band, then the film can form an unorientable surface (figure 1B). Two parallel rings not too far apart support a catenoid (figure 1C). The same two rings can also support a film which is a catenoid with a central disk sewn in (figure 1D). This film is no longer a manifold; it has a singular curve where three films meet. On a tetrahedral frame, a film forms that has a singular point at the center where six films and four triple curves meet (figure 1E). If the frame is a trefoil knot, it is possible to form a film that touches only part of the knot (figure 1F). This film was discovered not too long ago [AF1],[AT], and it was initially doubted that this film could exist with an ideal zero-diameter wire. The structure of the film where the wire leaves it was only elucidated recently [B2]. This knot film is a prime example that surprises still await us in soap films.

§2.2. Previous models. An early model of soap films was to represent a film as the graph of a function on a two-dimensional domain. This gave rise to the famous minimal surface equation, and enjoys all the theory developed for elliptic partial differential equations, but the range of phenomena covered is extremely limited. If the projection of the boundary curve on the plane is not convex, then the film may not be the graph of a function, as seen in the skew quadrilateral boundary in figure 1A. This model works for none of the examples of figure 1.
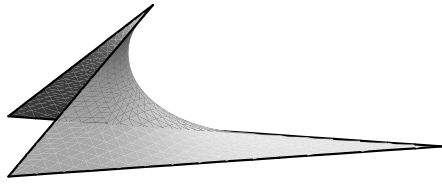
A. Skew quadrilateral.
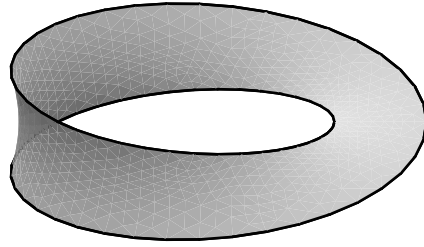
B. Mobius band.
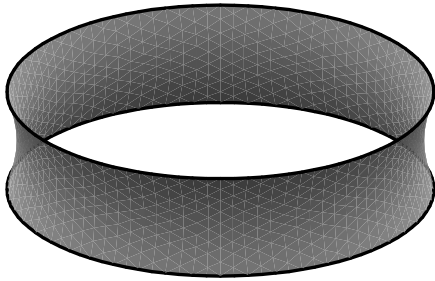
C. Catenoid.
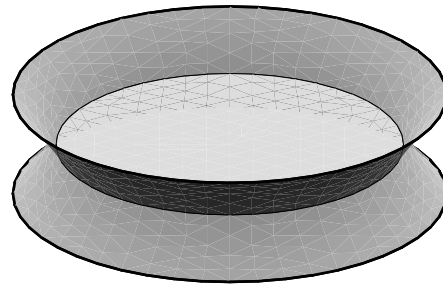
D. Catenoid with disk.
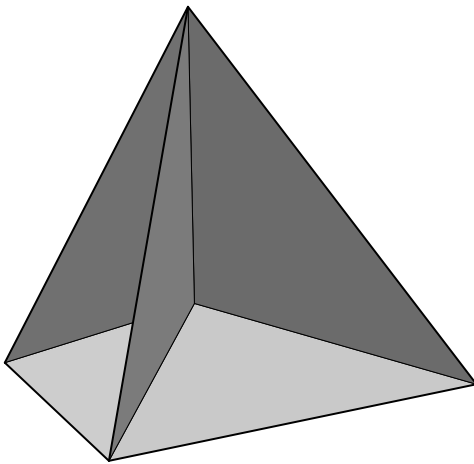
E. Tetrahedral film.
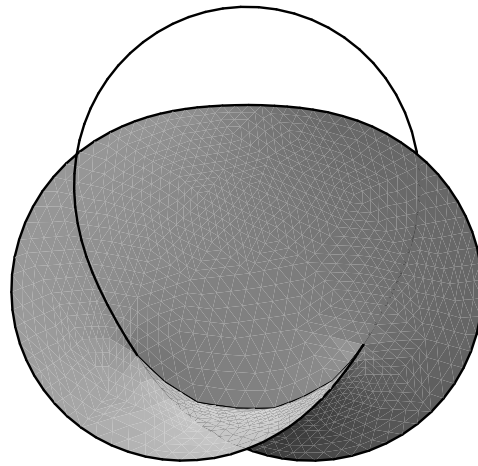
F. Trefoil knot film.

Figure 1. Various soap film examples. (Section 2.1)

Another classical approach is to represent a film as an image of a two-dimensional domain mapped into three-dimensional space. The famous Weierstrass representation of minimal surfaces is an example. Douglas [DJ] and Rado [RT] proved the existence of minimal surfaces that are topological disks bounded by given Jordan

curves. This type of approach requires a priori assumptions on the topological type of the surface. At best, it handles surfaces with singular curves and points very awkwardly, by stitching pieces together along their edges.

With the rise of measure theory, soap films were represented as rectifiable sets with Hausdorff measure [RE]. This eliminated the dependence on an a priori domain and permitted the representation of singularities, but rectifiable sets are rather intractable in regard to the fundamental problems mentioned above.

The integral current representation [FF],[F1] models surfaces as domains of integration of differential forms. It has the advantage of bringing to bear all the power of geometric integration theory, including a nice notion of boundary and calibration techniques. There is also a compactness theorem that gives existence. However, integral currents are oriented geometric objects, while soap films are not. Soap films may be unorientable and may have three films meeting along a curve. Integral currents mod ν attempt to cope with this by declaring currents with multiplicity ν equivalent to zero. Integral currents mod 2 can represent the Mobius band, and integral currents mod 3 can handle the catenoid with disk, but neither can handle the tetrahedral film or the trefoil knot film.

Almgren's varifolds [AF1] dropped the orientability of currents but kept formal tangent planes. A k-dimensional *varifold* is a Radon measure on the bundle of all unoriented tangent k-planes of a domain. Like rectifiable sets, they can represent all soap films, and like currents they solve the existence problem very well, having nice compactness properties. But the other fundamental problems are difficult.

Almgren's $(\mathbf{F}, \varepsilon, \delta)$-minimal sets (specializing to $(\mathbf{M}, 0, \delta)$-minimizing sets in the case of soap films) are a revival of rectifiable sets [AF2]. A set is $(\mathbf{M}, 0, \delta)$-*minimal* if its mass $\mathbf{M}$ (Hausdorff measure) cannot be decreased by small Lipschitz deformations. They have been used successfully with varifolds in existence and structure theorems. Taylor [TJ] showed that soap-film-like surfaces in three dimensions have only the singularities seen in the examples: triple curves and tetrahedral points. As with rectifiable sets, the fundamental problems of construction and proving minimality are difficult.

3. Preliminaries

Here we provide a brief reminder of the basic facts about differential forms and currents. Readers unfamiliar with forms and currents and wishing to get to the covering space model quickly may skip this section, and mentally think of currents and flat chains as oriented manifolds (not necessarily smooth) and of forms as the appropriate vectorfields to integrate over them.

A general reference for most of this material is [F1, §4.1]. The formulation here is slightly different for our later convenience. Usually forms and currents are done on smooth manifolds, but the ones we will use map naturally with Lipschitz maps, so we will be working in the Lipschitz category. But all our example manifolds will be smooth, so you may think of smooth manifolds if you wish.

§3.1. Domain. The domains of forms and currents will be Lipshitz Riemannian manifolds with boundary, generically denoted Y in this section. In a Lipschitz manifold with boundary, every interior point has a neighborhood lipeomorphic to an open ball and every boundary point has a neighborhood lipeomorphic to a

hemisphere. In even more generality, one could let Y be a Lipschitz space, in which every point has a neighborhood lipeomorphic to a Lipschitz neighborhood retract in some $\mathbb{R}^n$, but our examples will not need that generality. The Riemannian metric is assumed since we want to talk about area. The *closed boundary* of Y will be the set of manifold boundary points of Y, denoted $\partial^c Y$. The *open boundary* or *metric completion boundary* of Y is the set of points in the metric completion of Y that are not in Y, that is, the equivalence classes of Cauchy sequences in Y that have no limit in Y. The open boundary is denoted $\partial^o Y$. The completed space will be denoted $\bar{Y}$. It is not always true that $\bar{Y}$ is a Lipschitz space. A counterexample is the covering space of a punctured disk with the origin being an infinite order branch point.

It will turn out that the open boundary will contain the soap film boundary, while the closed boundary will act as source and sink for flows.

A compact Lipschitz space can be embedded as a whole in some Euclidean space, but we will use that embedding only to prove some compactness theorems. Otherwise our spaces exist in and of themselves without the need for extraneous dimensions.

§3.2. Forms and Currents.

Currents and differential forms are based on the exterior algebra of k-vectors $\bigwedge_k TY$ and k-covectors $\bigwedge^k TY$ on the tangent bundle TY of Y. A bit of surface is a k-vector and a bit of flow is a k-covector. A k-vector is *simple* if it is the exterior product of 1-vectors. A simple k-vector w has a well-defined k-dimensional area $|w|$. In $\mathbb{R}^{m+1}$, all m-vectors are simple, so one need not worry about the distinction between simple and nonsimple m-vectors when discussing soap films. The basic norm used here on general k-vectors is the *mass norm*:

$$||v|| = \inf \left\{ \sum_i |w_i| : w_i \text{ simple}, v = \sum_i w_i \right\}.$$

The dual *comass norm* on k-covectors is

$$||\omega|| = \sup\{\langle \omega, v \rangle : ||v|| \leq 1\}.$$

The space of *test k-forms* $\mathcal{D}^k(Y)$ consists of Lipschitz (C^∞ for smooth Y) k-covector valued functions with compact support in Y. Note that test forms must be zero near the open boundary of Y, but may be nonzero on the closed boundary. The space has the topology generated by the family of seminorms

$$\nu_K^i(\omega) = \sup\{||D^j\omega(y)|| : 0 \leq j \leq i, y \in K\}$$

where K ranges over all compact subsets of Y and i ranges over $\{0, 1\}$ (all nonnegative integers for smooth manifolds). Thanks to the duality between m-forms and 1-vectors provided by the metric in an $m+1$-dimensional Riemannian manifold, an m-form may be regarded as the velocity vectorfield of a flow.

A current is a domain of integration (in a very general sense) for forms. The space of k-*currents* $\mathcal{D}_k(Y)$ is the space of all continuous real-valued linear functions on $\mathcal{D}^k(Y)$. The integration of a k-form λ over a k-current T will be denoted $\langle T, \lambda \rangle$. Currents need not have compact support, so may extend all the way to the open boundary, which our soap films will do.

A k-current T and a p-form ϕ, with $k \geq p$, may be contracted by

$$\langle T \mathbin{\text{\Large$\llcorner$}} \phi, \lambda \rangle = \langle T, \phi \wedge \lambda \rangle.$$

The *comass norm* for $\omega \in \mathcal{D}^k(Y)$ is defined as

$$\mathbf{M}(\omega) = \sup\{\|\omega(y)\| : y \in Y\},$$

and the dual *mass norm* for $T \in \mathcal{D}_k(Y)$ is

$$\mathbf{M}(T) = \sup\{\langle T, \omega \rangle : \omega \in \mathcal{D}^k(Y), \mathbf{M}(\omega) \leq 1\}.$$

The mass of T is its generalized area. Mass (and other norms to be defined) may take on infinite values. Note that if a real-valued linear function on $\mathcal{D}^k(Y)$ has finite mass, then it is continuous and hence a current.

The standard boundary operator on currents and the exterior derivative on forms are dual operators in smooth manifolds:

$$\partial : \mathcal{D}_k(Y) \to \mathcal{D}_{k-1}(Y), \quad d : \mathcal{D}^{k-1}(Y) \to \mathcal{D}^k(Y), \quad \langle \partial T, \omega \rangle = \langle T, d\omega \rangle.$$

Not all currents have well-defined boundaries in Lipschitz spaces, though. But we will be working with a subspace of currents for which boundaries are well-defined. Intuitively, current boundary on the open boundary of Y does not count, but it does on the closed boundary. A closed m-form ($d\omega = 0$) corresponds to a divergenceless vectorfield, i. e. a conservative flow. If $f : Y_1 \to Y_2$ is a Lipschitz (or smooth) map, then there are induced dual mappings of currents and forms:

$$f_\# : \mathcal{D}_k(Y_1) \to \mathcal{D}_k(Y_2), \quad f^\# : \mathcal{D}^k(Y_2) \to \mathcal{D}^k(Y_1), \quad \langle f_\# T, \omega \rangle = \langle T, f^\# \omega \rangle.$$

A current $T \in \mathcal{D}_k(Y)$ is *representable by integration* if there is a Radon measure $\|T\|$ on Y and a $\|T\|$-measurable k-vector valued function $\vec{T}(y)$ on Y such that $\|\vec{T}(y)\| = 1$ for $\|T\|$-almost all $y \in Y$ and

$$\langle T, \omega \rangle = \int_Y \langle \vec{T}(y), \omega(y) \rangle d\|T\|y \qquad \text{for all } \omega \in \mathcal{D}^k(Y).$$

We can write $T = \|T\| \mathbin{\text{\Large$\llcorner$}} \vec{T}$. All currents of finite mass are representable by integration.

An oriented k-dimensional rectifiable set R has associated to it a natural k-current $[[R]]$. A *rectifiable k-current* is one which may be written as a countable sum of such $[[R]]$'s with finite total mass. The space of all rectifiable k-currents is denoted $\mathcal{R}_k(Y)$. By [F1,4.1.28(4)], a rectifiable current $R \in \mathcal{R}_k(Y)$ is associated to a rectifiable set $|R|$ in the sense that R can be represented

$$R = (\mathcal{H}^k \mathbin{\text{\Large$\llcorner$}} |R|) \mathbin{\text{\Large$\llcorner$}} \vec{R}$$

where $\vec{R}$ is the geometric tangent plane of $|R|$ and $|\vec{R}|$ is a positive integer $\mathcal{H}^k$-a.e on $|R|$.

An *integral k-current* is a rectifiable current whose boundary is also a rectifiable current. The space of integral k-currents is denoted $\mathbf{I}_k(Y)$.

The elements of $\mathcal{D}_k(Y)$ may be called *real currents* when the distinction between real and integral is being emphasized.

§**3.3. Flat chains and cochains.**   The *normal norm* on currents is

$$\mathbf{N}(T) = \mathbf{M}(T) + \mathbf{M}(\partial T)$$

and the dual norm on forms is

$$\mathbf{N}(\omega) = \inf\{\sup\{\mathbf{M}(\mu), \mathbf{M}(\nu)\} : \omega = \mu + d\nu\}.$$

The space of *normal k-currents* is

$$\mathbf{N}_k(Y) = \{T \in \mathcal{D}_k(Y) : \mathbf{N}(T) < \infty\}.$$

The *flat norm* on forms is

$$\mathbf{F}(\omega) = \sup\{\mathbf{M}(\omega), \mathbf{M}(d\omega)\}.$$

The dual flat norm on currents is

$$\begin{aligned}
\mathbf{F}(T) &= \sup\{\langle T, \omega\rangle : \omega \in \mathcal{D}^k(Y), \mathbf{F}(\omega) \le 1\} \\
&= \inf\{\mathbf{M}(Q) + \mathbf{M}(R) : Q \in \mathcal{D}_k(Y), R \in \mathcal{D}_{k+1}(Y), T = Q + \partial R\}.
\end{aligned}$$

The infimum is actually achieved. The space of *flat chains* $\mathbf{F}_k(Y)$ is the $\mathbf{F}$ closure in $\mathcal{D}_k(Y)$ of the normal currents $\mathbf{N}_k(Y)$. Clearly the boundary of a flat chain is also a flat chain. Not every current with finite flat norm is a flat chain; for example, a delta-function-like current has finite mass and hence finite flat norm, but is not a flat chain. The space of flat k-chains with finite mass will be denoted $\mathbf{FM}_k(Y)$. It is possible to characterize which currents are flat chains in terms of how they behave on smooth forms:

Proposition 3.1. *Suppose* $T \in \mathcal{D}_k(Y)$. *Then* $T \in \mathbf{F}_k(Y)$ *if and only if whenever*

  (1) $\omega_i \in \mathcal{D}^k(Y)$ *for* $i = 1, 2, \ldots,$
  (2) $\{\mathbf{F}(\omega_i)\}$ *is bounded, and*
  (3) $\lim_{i\to\infty} \mathbf{N}(\omega_i) = 0,$

then

$$\lim_{i\to\infty} \langle T, \omega_i\rangle = 0.$$

  □

The space of *integral flat chains* $\mathcal{F}_k(Y)$ is the flat closure of the integral currents $\mathbf{I}_k(Y)$. It includes the space of rectifiable currents $\mathcal{R}_k(Y)$. By [F1,4.2.16],

$$\begin{aligned}
\mathcal{R}_k(Y) &= \{T \in \mathcal{F}_k(Y) : \mathbf{M}(T) < \infty\}, \\
\mathbf{I}_k(Y) &= \{T \in \mathcal{R}_k(Y) : \mathbf{M}(\partial T) < \infty\}.
\end{aligned}$$

We will be able to work with integral currents since we will always have bounds on the mass and the boundary mass.

The space of *flat cochains* $\mathbf{F}^k(Y)$ is the dual space in the flat norm of $\mathbf{F}_k(Y)$. Remarkably, a flat cochain is equivalent to a Lebesgue measurable k-form ω for which $d\omega$ (defined weakly) is representable by a Lebesgue measurable form. If ω is not continuous, then the representation-by-integration formula does not apply directly. Rather, one takes the limit of integrals of smoothed versions of ω. Proposition 3.1 ensures that this is well-defined.

A map $f : Y_1 \to Y_2$ need only be Lipschitz for

$$f_\# : \mathbf{F}_k(Y_1) \to \mathbf{F}_k(Y_2), \quad f^\# : \mathbf{F}^k(Y_2) \to \mathbf{F}^k(Y_1)$$

to be well-defined. Hence, when considering only flat chains and cochains in a smooth manifold, only the Lipschitz structure of the manifold is significant. Thus in Lipschitz spaces, flat chains have well-defined boundaries, and flat cochains have well-defined exterior derivatives.

§3.4. **Homological equivalence.** We will be minimizing among homologous films. We will use two types of homology:

Definition 3.2. *Two currents* $S, T \in \mathcal{D}_k(Y)$ *are weakly homologous iff there is a* $Q \in \mathcal{D}_{k+1}(Y)$ *with* $S - T = \partial Q$. *Weak homology will be denoted* $S \sim T$.

Definition 3.3. *Two flat chains* $S, T \in \mathbf{F}_k(Y)$ *are flat homologous iff there is a* $Q \in \mathbf{F}_{k+1}(Y)$ *with* $S - T = \partial Q$. *Flat homology will be denoted* $S \approx T$.

Note that flat homologous chains are always weakly homologous, but the converse is not true. For example, if Y is the strip $\{(x, y) : 0 < y < 1\}$ and S is the oriented line segment from $(0, 1)$ to $(0, 0)$, then S is weakly homologous to the zero current since it bounds the right half-strip, but it is not flat homologous to zero since the flat norm of the right half-strip is infinite.

§3.5. **Compactness.** The fundamental compactness theorem for flat chains requires bounds on the mass of both the chains and their boundaries. Both real flat chains and integral currents have compactness. For Lipschitz spaces, the theorem follows from the compactness theorem proved for compact Lipschitz neighborhood retracts in [F1,4.2.17].

Theorem 3.4(Compactness). *If* Y *is a compact Lipschitz space and* $0 \leq c < \infty$ *then*

$$\{T \in \mathbf{F}_k(Y) : \mathbf{N}(T) \leq c\} \text{ is } \mathbf{F} \text{ compact, and}$$
$$\{T \in \mathbf{I}_k(Y) : \mathbf{N}(T) \leq c\} \text{ is } \mathbf{F} \text{ compact.}$$

□

We will also need a real version of an isoperimetric inequality.

Theorem 3.5[F2,3.1]. *If* Y *is a compact Lipschitz space, then there is a constant* $\alpha < \infty$ *such that if* $T \in \mathbf{F}_k(Y)$ *and* $\partial T = 0$ *then there is a* $Q \in \mathbf{F}_{k+1}(Y)$ *with* $\partial Q = T$ *and* $\mathbf{M}(Q) \leq \alpha \mathbf{M}(T)$.

□

Minimal films may not exist if the domain is unbounded in the wrong way. The following definition will characterize those spaces for which our existence proofs will work.

Definition 3.6. *A space Y is flat-k-bounded if there is a constant κ such that if $T \in \mathbf{F}_k(Y)$ and $T = \partial Q$ for some $Q \in \mathcal{D}_{k+1}(Y)$, then there is a $R \in \mathbf{F}_{k+1}(Y)$ with $T = \partial R$ and*

$$\mathbf{F}(R) \leq \kappa \mathbf{F}(T).$$

The key property of flat-k-bounded spaces for our purposes is that weak homology implies flat homology for bounded mass currents:

Proposition 3.7. *If Y is flat-k-bounded, $S, T \in \mathbf{F}_k(Y)$ and $S \sim T$, then $S \approx T$.*
□

Flat boundedness is not implied by ordinary boundedness. Consider the example of $Y = \{(x, y) : -1 \leq x \leq 1,\ 0 < y \leq 1\}$ with the metric

$$ds^2 = \frac{dx^2}{(x^2 + y^2)^2} + dy^2.$$

Then Y has infinite area, but finite diameter since a path from (x_1, y_1) to (x_2, y_2) that consists of three straight segments with vertices at $(x_1, 1)$ and $(x_2, 1)$ has length less than $\pi + 2$. Let $T_i \in \mathbf{F}_1(Y)$ be the upwardly oriented segment from $(1/i, 0)$ to $(1/i, 1)$. Then $T_1 - T_i$ has $\mathbf{F}(T_1 - T_i) \leq \mathbf{M}(T_1) + \mathbf{M}(T_i) = 2$ and bounds finite area, but the area increases without limit as $i \to \infty$. Hence there is no κ. The weak limit of the T_i is the upwardly oriented segment T_∞ from $(0, 0)$ to $(0, 1)$, but T_∞ is not flat homologous to any T_i. If the upper boundary of Y is modified to, say, $y = 1 - x^2/4$, then there is no minimal flat chain flat homologous to T_1.

Existence of minimal films will use compactness, so we make the following definition:

Definition 3.8. *A space Y is flat-k-compact if for any $c > 0$ the sets of flat chains*

$$G_c = \{R \in \mathbf{F}_k(Y) : \mathbf{N}(R) \leq c\} \text{ and } \mathcal{G}_c = \{R \in \mathbf{I}_k(Y) : \mathbf{N}(R) \leq c\}$$

are compact in the flat topology.

Remark. Flat-k-boundedness does not imply flat-k-compactness. The infinite strip $\{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq 1\}$ is flat-1-bounded but not flat-1-compact. I do not know if flat-k-compactness implies flat-k-bounded. Flat-k-compactness does not imply flat-$(k-1)$-compactness. I do not know if it implies flat-$(k+1)$-compactness.

We will want to have compactness only assuming $\bar{Y}$ is compact, not Y. This can be done using the quotient relationship between chains of finite mass of Y and $\bar{Y}$:

Proposition 3.9. *If $\bar{Y}$ is a compact Lipschitz space and $\partial^o Y$ is a compact Lipschitz neighborhood retract, then*

(1) $\mathbf{FM}_k(Y) \equiv \mathbf{FM}_k(\bar{Y})/\mathbf{FM}_k(\partial^o Y)$.
(2) $\mathbf{F}^k(Y) = \{\lambda \in \mathbf{F}^k(\bar{Y}) : \langle R, \lambda \rangle = 0 \text{ for } R \in \mathbf{F}_k(\partial^o Y)\}$.
(3) Y is flat-k-compact.
(4) Y is flat-k-bounded.

Remarks. Part (1) says that chains on the open boundary of Y don't count as chains in Y. In particular, a chain in $\bar{Y}$ whose boundary is in $\partial^o Y$ has no boundary

as a chain in Y. The mass finiteness is necessary in (1) since one could have Y be an open disk and take a 1-chain R to be a spiral approaching the circumference with infinite mass yet being a flat chain in Y but not $\bar{Y}$.

Part (2) states that flat cochains in Y have no flux out the open boundary. Together, parts (1) and (2) make open boundary behave appropriately as a wire or wall to anchor the edge of a soap film.

Part (3) gives us compactness for chains in Y, even when Y itself is not compact, and part (4) likewise says Y has isoperimetric behavior as if it were compact. These properties are sufficient for the existence proofs in this paper.

Proof. Let $g : U \to \partial^c Y$ be a Lipschitz neighborhood retract with domain a neighborhood U of $\partial^o Y$. By compactness, there is a minimum radius ρ of U around $\partial^o Y$.

(1) Suppose $R \in \mathbf{FM}_k(Y)$. R has finite mass, so it is representable by integration. Hence R is a current in $\bar{Y}$. Clearly, away from $\partial^o Y$ R is locally a flat chain in $\bar{Y}$. Let $f : \bar{Y} \to \mathbb{R}$ be the distance function from $\partial^o Y$. Clearly, f is Lipschitz. By the theory of chain slicing, [F1,4.3], for any $0 < \varepsilon < \rho$ one can find $0 < \delta < \varepsilon$ to divide R with the δ-level set of f into two pieces $R^+ = R \mathop{\llcorner} \{x : f(x) > \delta\}$ and $R^- = R \mathop{\llcorner} \{x : f(x) \le \delta\}$ such that the new boundary mass introduced on each is less than $\mathbf{M}(R)/\varepsilon$. Then $\mathbf{M}(R_\delta^-) \to 0$, so the $R_\delta^+ \in \mathbf{F}_k(\bar{Y})$ form an $\mathbf{F}$-Cauchy sequence converging to R. Hence $R \in \mathbf{FM}_k(\bar{Y})$.

Conversely, it is clear that an equivalence class of $\mathbf{FM}_k(\bar{Y})/\mathbf{FM}_k(\partial^o Y)$ gives a member of $\mathbf{FM}_k(Y)$.

(2) This follows as the dual of (1).

(3) Suppose $\{R_i\}$ is a sequence from G_c. The problem is to get control of the mass of the boundary of R_i after it is embedded in $\bar{Y}$. Much as in (1), we can slice each R_i to get a R_i^- with $spt(R_i^-) \subset U$ and new boundary mass under $\mathbf{M}(R_i)/r$. Let $\hat{R}_i = R_i + g_\# R_i^-$. Then

$$\mathbf{M}(R_i) \le \mathbf{M}(R_i) + Lip(g)^k \mathbf{M}(R_i^-) \le (1 + Lip(g)^k)c,$$

$$\mathbf{M}(\partial R_i) \le \mathbf{M}_Y(\partial R_i) + Lip(g)^{k-1}(g)\mathbf{M}(R)/\varepsilon \le (1 + Lip(g)^{k-1}/\varepsilon)c.$$

Hence by Theorem 3.2 there is a cluster point $R \in \mathbf{F}_k(\bar{Y})$, which is also a cluster point in $\mathbf{F}_k(Y)$. The same argument works with integral R_i.

(4) This follows from [F2,3.1]. $\qquad\square$

Proposition 3.10. *If Y is flat-k-compact, then $\mathcal{D}^k(Y)$ is $\mathbf{N}$-dense in $\mathbf{F}^k(Y)$.*

Proof. Suppose $\lambda \in \mathbf{F}^k(Y)$ but λ is not in the $\mathbf{N}$ closure of $\mathcal{D}^k(Y)$. Then there is an $\varepsilon > 0$ such that $\mathbf{N}(\lambda - \phi) > \varepsilon$ for all $\phi \in \mathcal{D}^k(Y)$. Let ϕ_i be an enumerated basis of $\mathcal{D}^k(Y)$. Then for each integer $j > 0$ there is a $Q_j \in \mathbf{F}_k(Y)$ with $\mathbf{N}(Q_j) \le 1$, $\langle Q_j, \lambda \rangle > \varepsilon$, and

$$\langle Q_j, \phi \rangle = 0 \text{ for } \phi \in span\{\phi_1, \dots, \phi_j\}.$$

By flat-k-compactness, there is a cluster point Q of the Q_i in the flat topology. Then $Q = 0$, since any $\phi \in \mathcal{D}^k(Y)$ is flat-close to some $span\{\phi_1, \dots, \phi_j\}$. But the flat limit also implies $\langle Q, \lambda \rangle > \varepsilon$, a contradiction. $\quad\square$

Applications of the Hahn-Banach Theorem will require a criterion for flat cochains to be flat chains.

Proposition 3.11. *If Y is flat-k-compact and T is a flat k-cocochain with $\mathbf{N}(T) < \infty$, then T is a flat k-chain.*

Proof. The restriction of T to $\mathcal{D}^k(Y)$ is a flat chain T_0 with $\mathbf{N}(T_0) \leq \mathbf{N}(T)$. The goal is to show $T = T_0$ as flat cocochains, which means showing

$$\langle T, \lambda \rangle = \langle T_0, \lambda \rangle \text{ for any } \lambda \in \mathbf{F}^k(Y).$$

By Proposition 3.10, for any $\varepsilon > 0$ there is $\lambda_\varepsilon \in \mathcal{D}^k(Y)$ with $\mathbf{N}(\lambda - \lambda_\varepsilon) < \varepsilon$. By definition of T_0,

$$\langle T, \lambda_\varepsilon \rangle = \langle T_0, \lambda_\varepsilon \rangle.$$

Hence

$$|\langle T - T_0, \lambda \rangle| = |\langle T - T_0, \lambda - \lambda_\varepsilon \rangle| < (\mathbf{N}(T) + \mathbf{N}(T_0))\mathbf{N}(\lambda - \lambda_\varepsilon) < 2\mathbf{N}(T)\varepsilon.$$

Hence

$$\langle T, \lambda \rangle = \langle T_0, \lambda \rangle.$$

$\square$

§3.6. Purely unflat currents.

Since we will be interested in flat chains, it will be convenient to be able to decompose currents into flat and unflat parts. Here we state the relevant definitions and properties.

Definition 3.12. *If $T \in \mathcal{D}_k(Y)$ and $\mathbf{M}(T) < \infty$, then T is a purely unflat current iff $\mathbf{M}(T - Q) \geq \mathbf{M}(T)$ for any flat $Q \in \mathbf{F}_k(Y)$.*

Remark. Intuitively, a purely unflat current is either too singular (like delta functions) or has its tangent k-vectors $\vec{T}$ unaligned with the geometric tangents to $\|T\|$.

Proposition 3.13. *If $T \in \mathcal{D}_k(Y)$ and $\mathbf{M}(T) < \infty$, then there is a decomposition $T = T_F + T_U$ such that*

    (1) $T_F \in \mathbf{F}_k(Y)$,
    (2) $T_U \in \mathcal{D}_k(Y)$ is purely unflat,
    (3) If $T \in \mathbf{F}_k(Y)$ then $T_U = 0$,
    (4) If $T \notin \mathbf{F}_k(Y)$ then $\mathbf{M}(T_U) > 0$ and $\mathbf{M}(T_F) < \mathbf{M}(T)$.

$\square$

Proposition 3.14. *If $\{T_i\}$ is a set of purely unflat k-currents such that $\sum_i \mathbf{M}(T_i) < \infty$, then $\sum_i T_i$ is also purely unflat.* $\square$

4. Motivating the covering space model

The goal of the covering space model is to represent any soap film as a flat chain, despite singularities. In fact, we will see that the singularities themselves may be represented in terms of flat chains. This section will motivate the covering space model by considering a concrete example, the triple junction. The next section will set up the formal apparatus.

§4.1. Example 4.1. The triple junction, or tripod. The concrete example is the least length set connecting three points which are the vertices of an equilateral triangle. This is a one-dimensional "film" which forms a triple junction at the center of the triangle, hence we shall call the film a *tripod*. It is shown at the top of figure 2, embedded in an ambient 2-manifold M. The basic idea is to represent the film as the boundary of its exterior. The exterior (shaded in figure 2) is the entire 2-chain $[\![M]\!]$. Unfortunately, $[\![M]\!]$ has no boundary where we want the film. Essentially, the two sides of the film have opposite orientation and hence cancel each other out. The solution is to realize that soap films are physically double surfaces with a very thin liquid layer between. It is natural to have each side of the film be the oriented boundary of the adjacent exterior region. But in the mathematical idealization of a zero-thickness film the two oppositely oriented sides cancel each other as chains. So the two sides need to be separated in order to prevent such cancellation. Re-introducing a non-zero thickness of internal liquid is not the answer, because at ideal equilibrium that liquid drains to zero thickness. One could cut the ambient space M along the film and add some new boundary to make a complete metric space with the film sides on the cut boundary. But that does not permit the variations of the film which characterize it as minimal.

The method used by the covering space model is to consider a covering space Y of M with the three vertices as branch points, and to regard the two sides of the film as living on different sheets of Y. For the tripod, the covering space we use is the triple cover of the complement of the three vertices. The three sheets, called A, B, and C, are cut and glued together in such a way that going around any vertex visits sheets A, B, and C in turn. Now the exterior of the film can be taken to be an integral 2-chain Q in Y which projects to $[\![M]\!]$. Q is shown as shaded in the second row of figure 2. The boundary of Q has two parts: the outer boundary of sheet A and the inner boundary forming the sides of the film. The outer boundary we denote S ; it is designated as part of the problem set-up and defines which sheet basically forms the exterior of the film. For later convenience, we take the orientation of S to be opposite that of ∂Q. The film is now represented as $R = \partial Q + S$. Note that the film sides are well separated.

The next step is to ensure that the film sides are indeed paired up. We create another covering space W of M that has one sheet for every pair of sheets of Y. It is shown in the third row of figure 2. A flat chain in sheet AB of W corresponds to a positive copy of itself in sheet A of Y and a negative copy in sheet B, and likewise for the other sheets of W. Now the tripod is represented by a 1-chain T in W composed of three segments from the center of the triangle to its vertices. Note that these segments have boundary at the center, where there was none in Y. This boundary is in fact the singular point of the tripod. It is the same 0-chain in all three sheets of W. To give this singularity a home, we make a covering space Z of M which has just one sheet, labelled ABC in the bottom row of figure 2. A flat chain here corresponds to positive copies in each of the sheets of W. The proper place to measure the mass of the film will turn out to be T in W, not R in Y. The mass of T will be called the *film-mass* of R. Section 5.8 below shows what can go wrong if the mass is measured in Y.
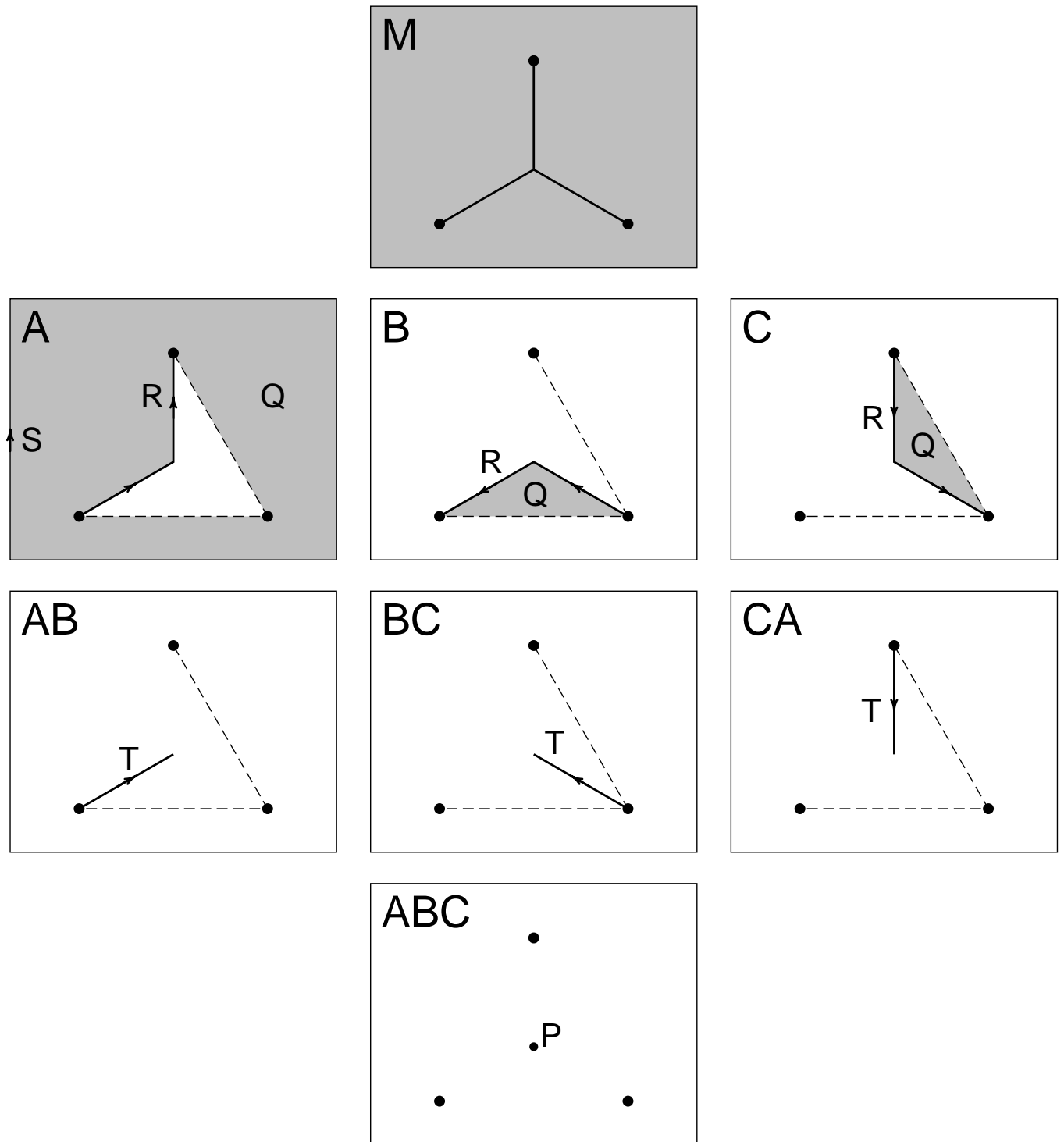
Figure 2. The representation of a tripod as a heirarchy of flat chains. The top picture shows the tripod and its exterior (shaded) in the base space M. The second row shows the three sheets of the covering space Y. The exterior lifts to the shaded region, whose boundary is the sides of the tripod film. The third row shows the pair covering space, with the pair representation of the film. The bottom row shows the first singular space, whose one-point 0-chain represents the triple point. The dashed lines show the cuts for gluing together the sheets of the covering spaces. (Section 4.1)

Once the gluing across the cut is done to construct the covering space, the exact location of the cut is immaterial. The tripod could be used as the cut. In fact, any cut will define a possible soap film, since any cut must separate sheet A from the others and hence provide a boundary for sheet A. Hence the tripod is the least length cut that could be used to construct this covering space. Proving that the tripod actually is the solution will be done in section 8.3 using calibration.

The film R has no homological boundary itself in Y, since $\partial\partial Q = 0$. Hence it is improper to try to think of the three vertices as the homological boundary of the film.

This soap film problem may be stated thus: among all integral 2-chains Q in Y that project to $[[M]]$, find the one whose inner boundary $R = \partial Q + S$ has minimal film-mass. Note that no assumptions are made on the topology of the final film.

5. COVERING SPACES

This section generalizes and formalizes the covering space model introduced in the previous section. It defines the various covering spaces and the mappings between them. It also shows how a film corresponds to a chain complex with the various dimensional skeletons in the different covering spaces.

§**5.1. Base manifold.**   The ambient space of a soap film will be an $(m + 1)$-dimensional Lipschitz Riemannian manifold-with-boundary E. In the examples below, E will be a subset of 2 or 3-dimensional Euclidean space. The film will be supported by a boundary set $B \subset E$. The term "supported" is just intuitive here; all we actually require is that B be a closed set. The boundary set is usually the wire frame (or other physical object) supporting the film. However, sometimes it will be necessary to introduce additional "invisible wires" to act as obstacles to the deformation of the film to a smaller area film. The film need not contact all of B.

The actual base manifold for covering spaces will be the complement of the boundary, $M = E - B$. The reason that the boundary set B is omitted from M is that a covering space of M generally does not extend to a covering space of E. For example, a nontrivial two-fold cover of the punctured plane does not extend to a two-fold cover of the whole plane, unless one includes a branch point. However, we will not include branch points in our covering spaces so that we may assume any point of M locally has a trivial cover.

Distances between points of M will always be the shortest-path distance within M. For example, if the boundary set B is a half-plane in $\mathbb{R}^3$, then the metric completion of its complement separates the two sides of the plane, which is proper since soap films on opposite sides don't affect each other directly.

We will assume M is orientable. If not, one may consider the base domain to be the oriented double cover and restrict all chains to be symmetric between the two sheets. The connectedness of M is not significant.

In choosing E and B, one should realize that the entire open boundary of M will be support for the film, and the closed boundary of M will in effect be an obstacle that the film may run up against.

§**5.2. Primary covering space.**   The domain of the flat chain representing the exterior of a soap film will be an $(m + 1)$-dimensional Lipschitz Riemannian

manifold-with-boundary Y that is a covering space of M with projection map p : $Y \to M$. Y will be called the *primary covering space* (this supersedes the generic use of Y in the Preliminaries section). When we have need to refer to the different sheets of Y, we will use alphabetical labels A, B, ... For precision, we will suppose that Y is made by gluing together a number of copies of M with one common set of cuts. Hence the sheet labellings are well-defined at any point. Of course, if one is considering a particular point x of M, then one may assume the cuts avoid a neighborhood of x.

The $(m+1)$-chain representing the exterior of the film lives in Y. It will be denoted by Q throughout this paper, and may be regarded as the $(m+1)$-dimensional skeleton of the film.

Choosing Y is part of the design of a soap film problem. Y need not be connected. In problems where the film is an interface between different fluids, one can use one sheet of Y for each fluid. This then becomes equivalent to the scheme of [LM] and [B1]. However, for soap film problems where the complement of the film is connected, Y will usually be connected. Nor is there a need for Y to have a finite number of sheets, although some of the existence theorems will require finiteness.

§5.3. Invisible wires. The boundary wire B in M becomes branch points (for 2D) or branch curves (for 3D) in $\bar{Y}$. The order of branching at a branch point is the number of sheets meeting at the point. In an n-sheeted covering space, a piece of boundary may lift to several distinct branch curves of total order n. The orders may be different; in a three-sheeted cover, a curve may lift to branch curves of order 1 and order 2. An order 1 branch curve is not really a singularity in the covering space; it is "invisible". A boundary wire that gives rise to an order 1 branch curve in the sheet that is the exterior of a soap film is called an "invisible wire". Invisible wires may arise naturally (examples 7.4,7.5) or it may be necessary to add them (examples 7.7, 7.8).

§5.4. The reference surface. To form the outer boundary of the exterior Q and to determine which sheet basically is the exterior of the film, a *reference surface* $S \in \mathbf{F}_m(Y)$ may be designated. The requirements on S are that $\partial S = 0$ and $p_\# S = -\partial[\![M]\!]$. The film itself is $R = \partial Q + S$. The orientation of S is chosen so that R has the same orientation as ∂Q and is homologous to S.

§5.5. Pair covering space. In order to represent double-sidedness of films and define film-mass, we will need a further covering space W of Y, called the *pair covering space*, which has sheets for ordered pairs of distinct sheets of Y. A given Y may have many possible W's. The choice of W is part of the design of the problem. First, we define the full pair covering space W_0 which contains all pairs:

$$W_0 = \{(y_1, y_2) \in Y \times Y : y_1 \neq y_2, p(y_1) = p(y_2)\}.$$

Let $q_1, q_2 : W_0 \to Y$ be projections defined by

$$q_1(y_1, y_2) = y_1, \quad q_2(y_1, y_2) = y_2.$$

The actual pair covering space used will consist of selected components of the full covering space W_0 (*component* in the topological sense). The aim is that only pairs

that can legitimately form opposite sides of a film are included in W. For example, if one wants the film to be without triple junctions, then one should construct Y and W so that there are no cyclic triples of pairs AB,BC,CA. However, we will assume that the pairs of W span all pairs in the sense that any pair of sheets of Y may be linked by a finite chain of pairs in W. If this is not the case, then the effect is equivalent to replacing M with a covering space of M for which Y is still a covering space and for which W does span all pairs of Y. The minimum number of pairs of W needed to span any pair of W_0 will be the *pair distance* of that pair. The maximum pair distance will be the *pair diameter* of W. It may be infinite.

Note that projections $q_1, q_2 : W \to Y$ give natural dual mappings of chains and cochains:

$$q_\# : \mathbf{F}_k(W) \to \mathbf{F}_k(Y) \text{ defined by } q_\# T = q_{1\#} T - q_{2\#} T,$$
$$q^\# : \mathbf{F}^k(Y) \to \mathbf{F}^k(W) \text{ defined by } q^\# \omega = q_1^\# \omega - q_2^\# \omega.$$

The first is how we represent double-sided films, and the second finds the differences of flows on different sheets. Flat chains in Y that are images under $q_\#$ will be called *double-sided flat chains*. Because of the spanning assumption on W, no double-sided chains are lost in going from W_0 to W; that is, $q_\# F_k(W) = q_\# F_k(W_0)$.

The film will be represented in W by a flat chain T such that $q_\# T = R$. T is the m-skeleton of the film chain complex, and is sometimes the chain best thought of as representing the film itself, although most of the time that honor should go to $R = \partial Q + S$ in Y. The *film-mass* of a soap film is the mass of T, not that of R, for reasons explained in section 5.8 below.

§**5.6. Singularity covering spaces.** We next describe the general pattern of extending the sequence of covering spaces to include those holding the chains representing singularities.

The covering spaces will form a finite sequence $Z_0, Z_1, \ldots, Z_j$. Let Z_i be the covering space of M which is the home of the $(m-i+2)$-dimensional skeleton chain of the film. It is not necessary that Z_{i+1} be a covering space of Z_i; the singularity covering space Z of example 4.1 shows this. We let Z_0 be M, with just one sheet. We let Z_1 be Y, with the sheet labels A,B,.... We let Z_2 be W, with the sheet labels being pairs of Y sheet labels. Usually there are many possible choices for each Z_i, and which is used is part of the design of the problem. It is not necessary to continue the construction beyond W if one does not care about representing singularities.

Suppose Z_i has sheets indexed with an index set I. Let Z_{i+1} be the next covering space in the sequence, with index set J. The connection between Z_{i+1} and Z_i is established by specifying locally for each $\alpha\beta \in I \times J$ a coefficient $a_{\alpha\beta} \in \{0, 1, -1\}$ in a globally continuous manner. There is a natural map

$$r_{i\#} : \mathbf{F}_k(Z_{i+1}) \to \mathbf{F}_k(Z_i)$$

such that if T_β is the part of current T on sheet β of Z_{i+1} then the part of the image on sheet α of Z_i is

$$(r_{i\#}T)_\alpha = \sum_{\beta \in J} a_{\alpha\beta} T_\beta.$$

The coefficients $a_{\alpha\beta}$ may be viewed as forming a matrix $H_i : \mathbb{R}^J \to \mathbb{R}^I$. In case I is infinite, we make the stipulation that H_i is bounded in the L_1 norm as a linear transformation. The L_1 norm of H_i will be denoted c_i. In case J is infinite, we require that H_i have a bounded right inverse defined on $image(H_i)$. This bound will be denoted d_i.

We make the stipulation that the sequence of H_i's is exact, that is, $image(H_i) = ker(H_{i-1})$. This implies that

$$\mathbf{F}_k(Z_{i+1}) \xrightarrow{r_{i\#}} \mathbf{F}_k(Z_i) \xrightarrow{r_{i-1\#}} \mathbf{F}_k(Z_{i-1})$$

is exact, that is, the image of $r_{i\#}$ is the kernel of $r_{i-1\#}$. The idea is that if $T \in \mathbf{F}_k(Z_i)$ is the k-skeleton of the film, then its boundary is the image under $r_{i\#}$ of the $(k-1)$-skeleton in $\mathbf{F}_{k-1}(Z_{i+1})$.

Definition 5.1. *A* suite of covering spaces *$\mathcal{Z}$ is a tuple $\{Z_0, \ldots, Z_j\}$ of covering spaces of M with maps r_i as just described.*

A suite of covering spaces forms a commutative diagram:

$$
\begin{array}{ccccccc}
 & & & & \mathbf{F}_{m+1}(Y) & \xrightarrow{p_\#} & \mathbf{F}_{m+1}(M) \\
 & & & & \downarrow{\partial} & & \downarrow{\partial} \\
 & & \mathbf{F}_m(W) & \xrightarrow{q_\#} & \mathbf{F}_m(Y) & \xrightarrow{p_\#} & \mathbf{F}_m(M) \\
 & & \downarrow{\partial} & & \downarrow{\partial} & & \\
 & \mathbf{F}_{m-1}(Z_3) & \xrightarrow{r_{2\#}} & \mathbf{F}_{m-1}(W) & \xrightarrow{q_\#} & \mathbf{F}_{m-1}(Y) & \\
 & & \downarrow{\partial} & & \downarrow{\partial} & & \\
\mathbf{F}_{m-2}(Z_4) & \xrightarrow{r_{3\#}} & \mathbf{F}_{m-2}(Z_3) & \xrightarrow{r_{2\#}} & \mathbf{F}_{m-2}(W) & & \\
\downarrow{\partial} & & \downarrow{\partial} & & & & \\
\mathbf{F}_{m-3}(Z_4) & \xrightarrow{r_{3\#}} & \mathbf{F}_{m-3}(Z_3) & & & & \\
\downarrow{\partial} & & \downarrow{\partial} & & & & \\
\cdots & & & & & &
\end{array}
$$

(Diagram 5.1)

Of course, there is a similar diagram for currents, and there is a dual diagram of cochain spaces.

The skeleton chain in Z_i will be denoted P_i with $P_1 = Q$, and $P_2 = T$. P_0 doesn't properly exist, since that would require $[\![M]\!] = \partial P_0$.

Definition 5.2. *A* film chain complex *$\mathcal{P}$ is a tuple $(P_1, \ldots, P_j)$ with $P_i \in \mathbf{F}_{m-i+2}(Z_i)$ such that*

$$r_{1\#}P_2 = \partial P_1 + S,$$
$$r_{i\#}P_{i+1} = \partial P_i \quad \text{for } i \geq 2.$$

The skeleton chains form a corresponding commutative diagram:

$$
\begin{array}{ccc}
Q & \xrightarrow{\ p_\# \ } & M \\[2pt]
\downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial}
\end{array}
$$

$$
\begin{array}{ccccc}
T & \xrightarrow{\ q_\# \ } & \partial Q & \xrightarrow{\ p_\# \ } & 0 \\[2pt]
\downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \\[2pt]
P_3 & \xrightarrow{\ r_\# \ } & \partial T & \xrightarrow{\ q_\# \ } & 0
\end{array}
$$

(Diagram 5.2)

$$
\begin{array}{ccc}
P_4 & \xrightarrow{\ s_\# \ } & \partial P_3 & \xrightarrow{\ r_\# \ } & 0 \\
\downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} \\
\partial P_4 & \xrightarrow{\ r_\# \ } & 0 \\
\downarrow{\scriptstyle \partial} \\
\cdots
\end{array}
$$

We have assumed $\partial M = 0$ so $S = 0$ and $R = \partial Q$ in order to make the diagram look neater. Problem formulations later will have to use a nonzero ∂M. The diagram sidesteps the question of which of many possible chains to take as one moves to the left in the diagram. Without some criterion like least mass, we can't get canonical representatives.

This diagram is the desired state of affairs. It remains to be shown that it can indeed be constructed in any particular case. Soap film problems do not need the entire diagram; everything beyond W is optional.

Unfortunately, this scheme does not capture all singular structure in higher dimensions, where the singular set can skip dimensions. For example, in $\mathbb{R}^8$ the cone over $S^3 \times S^3$ is a minimal chain which has just one singular point at the origin [BDG]. Is there some profound meaning to be found in the difference between singular structure that shows up in the homology and that which doesn't?

§**5.7.   Film-mass.**   Now we consider the proper way to measure the mass of a soap film. We have seen that a film may be viewed as $T \in F_m(W)$ or as $R = \partial Q + S \in F_m(Y)$ with $q_\# T = R$. We cannot simply take $\mathbf{M}(R)$ to be the mass of the film due to the problems discussed below in section 5.8, so we turn to T. In general, there are many T of different masses with $q_\# T = R$. We will define the film-mass of the film as the least such mass. It is convenient to start with film-comass.

Definition 5.3. *The film-comass of a k-form $\omega \in \mathbf{F}^k(Z_i)$ is*

$$
\mathbf{M}^\dagger(\omega) = \mathbf{M}(r_i^\# \omega).
$$

Note that film-comass is only a seminorm. In the most common case $Z_i = Y$, the film-comass is the bound on the difference of ω on pairs of sheets, but only those pairs that are included in W.

Definition 5.4. *The film-mass of a k-current $R \in \mathcal{D}_k(Z_i)$ is*

$$\mathbf{M}^\dagger(R) = \sup\{\langle R, \omega \rangle : \omega \in \mathcal{D}^k(Z_i), \mathbf{M}^\dagger(\omega) \le 1\}.$$

Note that film-mass in Y is finite only for double-sided currents.

Definition. *Suppose $R \in \mathcal{D}_k(Z_i)$. Then $T \in \mathcal{D}_k(Z_{i+1})$ is a minimal film-mass representative of R if $r_{i\#}T = R$ and $\mathbf{M}(T) = \mathbf{M}^\dagger(R)$.*

In general, T is not unique. This is particularly easy to see when W contains both sheets AB and BA, since T may be apportioned arbitrarily between them.

Proposition 5.5. *If $R \in \mathcal{D}_k(Z_i)$ then*

$$\mathbf{M}^\dagger(R) = \inf\{\mathbf{M}(T) : T \in \mathcal{D}_k(Z_{i+1}), R = r_{i\#}T\}.$$

Further, if $\mathbf{M}^\dagger(R) < \infty$, then R has a minimal film-mass representative T.

Proof. If $R = r_{i\#}T$ and $\mathbf{M}^\dagger(\omega) \le 1$, then

$$\begin{aligned}
\langle R, \omega \rangle &= \langle r_{i\#}T, \omega \rangle \\
&= \langle T, r_i^\# \omega \rangle \\
&\le \mathbf{M}(T).
\end{aligned}$$

Thus

$$\mathbf{M}^\dagger(R) \le \inf\{\mathbf{M}(T) : T \in \mathcal{D}_k(Z_{i+1}), R = r_{i\#}T\}.$$

In the case that $\mathbf{M}^\dagger(R) < \infty$, equality will be shown by constructing a T whose mass is at most $\mathbf{M}^\dagger(R)$. Consider the linear space $\mathcal{D}^k(Z_{i+1})$ with the mass norm. On the subspace $r_i^\# \mathcal{D}^k(Z_i)$ of $\mathcal{D}^k(Z_{i+1})$, define the linear function

$$L(r_i^\# \omega) = \langle R, \omega \rangle,$$

and note that

$$L(r_i^\# \omega) \le \mathbf{M}(r_i^\# \omega)\mathbf{M}^\dagger(R).$$

The Hahn-Banach Theorem extends L to a linear function on all of $\mathcal{D}^k(Z_{i+1})$ such that

$$L(\lambda) \le \mathbf{M}(\lambda)\mathbf{M}^\dagger(R).$$

Define $T \in \mathcal{D}_k(Z_{i+1})$ by

$$\langle T, \lambda \rangle = L(\lambda) \qquad \text{for } \lambda \in \mathcal{D}^k(Z_{i+1}).$$

Then for $\omega \in \mathcal{D}^k(Z_i)$,

$$\langle R, \omega \rangle = L(r_i^\# \omega) = \langle T, r_i^\# \omega \rangle = \langle r_{i\#}T, \omega \rangle,$$

so $R = r_{i\#}T$. Finally,

$$\mathbf{M}(T) = \sup\{\langle P, \lambda\rangle : \mathbf{M}(\lambda) \le 1\}$$
$$= \sup\{L(\lambda) : \mathbf{M}(\lambda) \le 1\}$$
$$\le \mathbf{M}^\dagger(R).$$

This mass bound also shows that T is indeed a current. Hence T is the desired minimal film-mass representative of R. $\square$

When $Z_i = Y$, film-mass measures how R may be decomposed into pairs of sides. Since not all pairs need be included in W, some pairs may need to be represented as telescoping sums of pairs that are in W. Hence it is possible for the film-mass to be much larger than the mass. However, mass and film-mass are equivalent norms on the currents of interest.

Proposition 5.6. *Let c_i be the L_1 bound of H_i and let d_i be the L_1 bound of the generalized right inverse of H_i. For $R \in r_{i\#}\mathcal{D}_k(Z_{i+1})$ we have*

$$\mathbf{M}(R) \le c_i\mathbf{M}^\dagger(R),$$
$$\mathbf{M}^\dagger(R) \le d_i\mathbf{M}(R).$$

For $\omega \in \mathbf{F}^k(Z_i)$

$$\mathbf{M}^\dagger(\omega) \le c_i\mathbf{M}(\omega),$$

and there is $\lambda \in F^k(Z_{i-1})$ such that

$$\mathbf{M}(\omega - r_{i-1}^\#\lambda) \le d_i\mathbf{M}^\dagger(\omega).$$

$\square$

In the common case of film-mass in Y, we can be more precise:

Proposition 5.7. *If $R \in \mathcal{D}_k(Y)$ then*

$$\mathbf{M}(R) \le 2\mathbf{M}^\dagger(R).$$

If W has pair diameter d and $R \in q_\#\mathbf{F}_k(W)$ then

$$\mathbf{M}^\dagger(R) \le d\mathbf{M}(R)$$

and the inequality is strict if $\mathbf{M}(R)$ is finite and nonzero.

Proof. The first inequality is clear, since each sheet of W maps to two sheets of Y. Mass is a local property, so the proof of the second inequality may be done in neighborhoods small enough so the covering is trivial. Suppose $R \in q_\#\mathbf{F}_k(W)$. Let R_α denote the part of R on sheet α. First, represent R in W_0 by picking a distinguished sheet, say $\alpha = 0$, and for $\beta \ne 0$ define

$$T_{\beta 0} = R_\beta.$$

Then $q_\# T = R$ (since $\sum R_\alpha = 0$ makes R_0 come out right) and $\mathbf{M}(T) \leq \mathbf{M}(R)$. If pair $\beta 0$ is spanned by the chain of pairs $\beta\sigma_1, \ldots, \sigma_d 0$ in W, then one can define

$$T_{\sigma_j, \sigma_{j+1}} = R_\beta.$$

So $\mathbf{M}(T) \leq d\mathbf{M}(R)$, hence $\mathbf{M}^\dagger(R) \leq d\mathbf{M}(R)$. If $0 < \mathbf{M}(R) < \infty$, then sheet 0 may be picked to have positive mass, and the inequality is strict. $\square$

Note the previous theorem only guarantees the existence of a current, not a flat chain. The problem is that the boundary of the representative is not under control. However, representatives can be proven flat by showing that any unflat part is unnecessary:

Proposition 5.8. *Suppose $R \in \mathbf{F}_k(Z_i)$ has a minimal film-mass representative $T \in \mathcal{D}_k(Z_{i+1})$. Then $T \in \mathbf{F}_k(Z_{i+1})$.*

Proof. Suppose $T \notin \mathbf{F}_k(Z_{i+1})$. Then by Proposition 3.13, there is a decomposition $T = T_F + T_U$ with flat T_F and purely unflat T_U with $\mathbf{M}(T_F) < \mathbf{M}(T)$. Then

$$r_{i\#} T_F + r_{i\#} T_U = R,$$

so

$$r_{i\#} T_U = R - r_{i\#} T_F.$$

The right side of this is flat, and the left side is purely unflat by Proposition 3.14, so $r_{i\#} T_U = 0$. Hence T_F alone is a representative of R, which contradicts T being the least mass representative. $\square$

In the rectifiable case, a representative need not be rectifiable since a linear combination of representatives need not have integer densities. However, it can be shown in the case of most interest that there is always a rectifiable representative:

Proposition 5.9. *Suppose $R \in \mathcal{R}_m(Y) \cap q_\# \mathbf{F}_m(W)$ has finite mass. Then R has a rectifiable minimal film-mass representative $T \in \mathcal{R}_m(W)$.*

Proof. By Propositions 5.5 and 5.8, R has a minimal film-mass representative $P \in \mathbf{F}_m(W)$. Both R and P have finite mass, so are representable by integration. Instead of the standard representation in terms of measures on Y and W, we shall consider representations over M. View $\bigwedge_m TY$ as a vector bundle over M with fiber $(\wedge_m TM)^n$, where n is the number of sheets in Y (possible infinite). Take $||R||$ to be a Radon measure on M and $\vec{R} \in (\wedge_m TM)^n$, so that $R = ||R|| \llcorner \vec{R}$. Likewise, let $(\wedge_m TM)^p$ be the fiber of W over M, with P represented as $||P|| \llcorner \vec{P}$. Since R is rectifiable, $||R||$ is Hausdorff measure restricted to a rectifiable set. Furthermore, there is a unique geometric tangent at $||R||$ almost all points, and all components of $\vec{R}$ are parallel to this tangent and have integral magnitude. $||P||$ is continuous with respect to $||R||$, since if $||P||$ had any singular measure, then the corresponding part of P could be excised, reducing the mass while keeping a representative of R. Since P is flat and $||P||$ is continuous with respect to $||R||$, all components of $\vec{P}$ must also be parallel to this direction. Revise the magnitudes of the components of $\vec{P}$ so that P is represented as $P = ||R|| \llcorner \vec{P}$. To construct a rectifiable version of P, it is only

necessary to get the components of $\vec{P}$ to have integral magnitude $||R||$-a.e. This can be done pointwise. Suppose a component of $\vec{P}$ has fractional density on some sheet, say sheet AB. Since the projection on Y has integral magnitudes, there must be a sheet BX for some X with fractional magnitude. One can continue in this manner until the chain of sheets closes up at AB again. Subtract the same magnitude from each sheet so that at least one sheet has integral magnitude. This does not alter the projection on Y, since each sheet (such as B) has the same amount added and subtracted. This process reduces the number of fractional sheets, and may be repeated until all the sheets are integral. This does not increase the total mass, because if it did then the opposite change would decrease the mass, contradicting the film-mass minimality of P. The ultimate result is a rectifiable T. □

Remark. This proof does not seem to carry over to higher covering spaces because one does not get the closed chain of sheets. Hence the proof would only give that the representatives are rectifiable currents times $||R||$-measurable real scalar functions.

Proposition 5.10. *Film-mass is lower semicontinuous in flat and weak topologies.*

Proof. It suffices to prove it in the case of weak topology, since the flat topology is stronger. Suppose $R, R_j \in \mathcal{D}_k(Z_i)$ and $R_j \to R$ weakly. Then for each $\omega \in \mathcal{D}^k(Z_i)$ with $\mathbf{M}^\dagger(\omega) \leq 1$,

$$\langle R, \omega \rangle = \lim_{j \to \infty} \langle R_j, \omega \rangle \leq \liminf_{j \to \infty} \mathbf{M}^\dagger(R_j),$$

so

$$\mathbf{M}^\dagger(R) \leq \liminf_{j \to \infty} \mathbf{M}^\dagger(R_j).$$

□

That film-mass is not continuous in the flat or weak topologies is shown by considering R_j to consist of oppositely oriented films that cancel in the limit.

§**5.8. A faulty formulation.** One part of the covering space model of soap films is to get the two surfaces on different sheets of Y. The second part is to get the two surfaces in the same place in M. It does not work to omit the pair space W and just require the soap film exterior to be an $m+1$-chain Q in Y that projects to $[[M]]$ and minimize the mass of the film in Y.

Example 5.11. Smeared tripod. A counterexample is a smeared tripod, shown in figure 3. Imagine a small equilateral triangle upside down at the center. As shown in figure 3, take the exterior Q to be a real 2-chain which is a smoothed version of the Q of figure 2. The film components $R = \partial Q + S$ in Y fan out from the one vertex to the center triangle, cross the triangle parallel to one side, and fan in to another vertex. Then Q projects to $[[M]]$, but the total mass of R is less than the tripod. But the tripod is supposed to be minimal. This apparent contradiction is resolved when one calculates the mass of the minimal film-mass current T in W that projects to R. That mass turns out to be greater than the tripod. This shows that we cannot dispose of W and work only in Y.

It does not work to add the condition that R be generated from chains that locally add to zero on pairs of sheets, because this example fits that condition. Thanks to Frank Morgan for pointing this out.
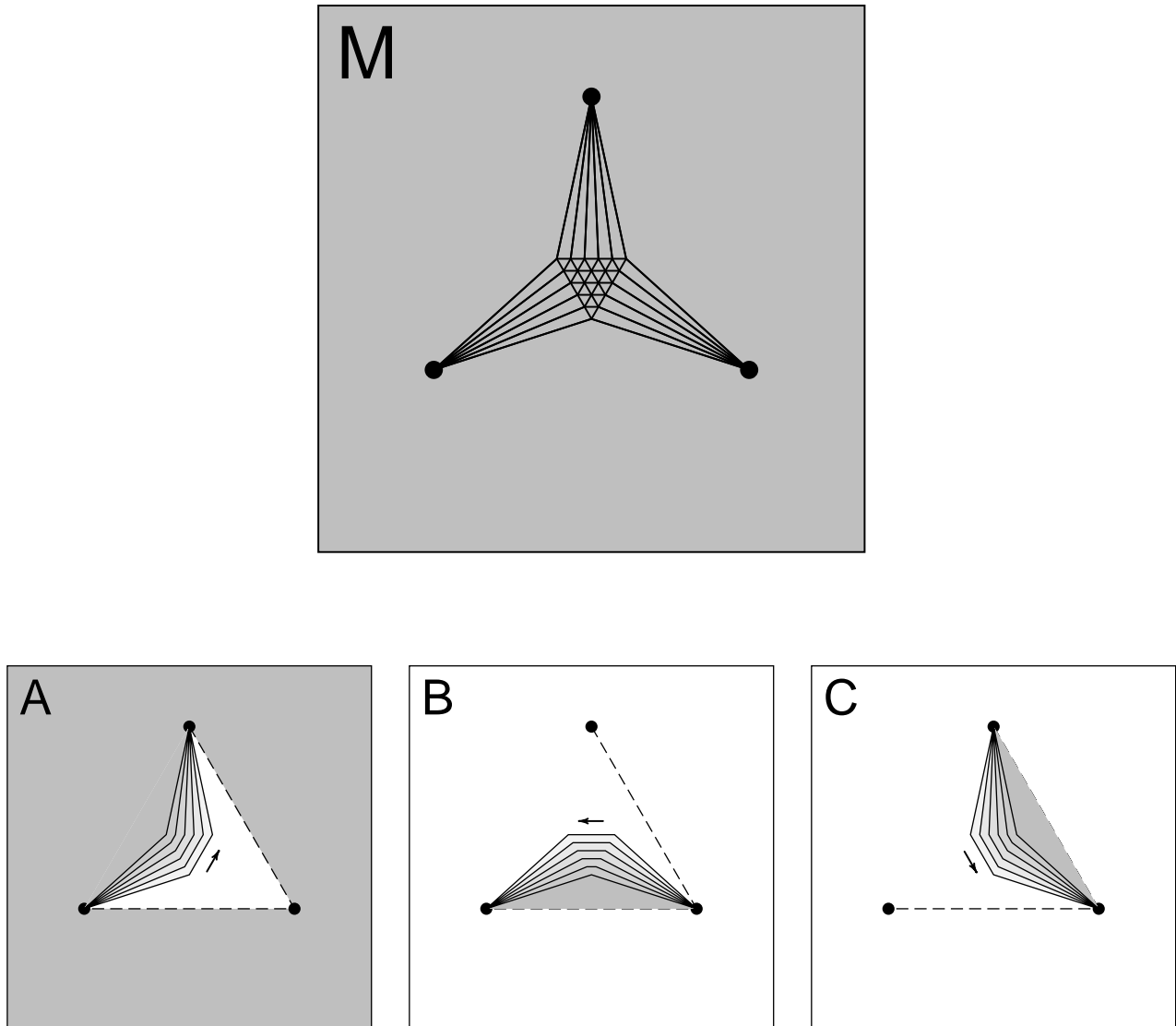
Figure 3. Smeared tripod joining three vertices of an equilateral triangle. Top picture shows the full smeared tripod in the base space. Second row shows the sheets of Y with the oriented smeared film. All these currents are actually continuous with respect to Lebesgue measure. (Section 5.8)

This kind of smearing does not happen with integral chains, but clearly the integral and real problems should be formulated in the same way. Also, the choice of W affects the definition of film-mass, and hence the solution, in the integral case.

6. Soap film problem statements and existence

Now we are in a position to state soap film minimization problems. There are several problems that can be stated. All of them have in common

M The base manifold, usually constructed by removing a closed boundary set B from a Riemannian manifold.

Y A primary covering space of M.
W A pair covering space of Y.
S A reference surface in Y.
Each of the problems is tagged with a label for ease of reference.

§**6.1. Integral soap film problem.** The goal is to find an integral current soap film. We frame the problem as finding an exterior Q whose boundary has minimal film-mass. We want Q to be a single cover of M, so we require Q to be the current corresponding to a measurable subset of Y with multiplicity $+1$. The problem is

$$
\begin{array}{ll}
\text{For} & Q \in \mathbf{I}_{m+1}(Y) \\
\text{minimize} & \Psi = \mathbf{M}^{\dagger}(\partial Q + S) \\
\text{such that} & Q = [\![X]\!] \text{ for some } X \subset Y \\
& p_{\#}Q = [\![M]\!].
\end{array}
\qquad \text{(IMIN)}
$$

Any solution Q has associated to it the integral m-chain $R = \partial Q + S$, which represents the sides of the film in Y, and possibly many minimal film-mass representatives $T \in \mathbf{F}_m(W)$ for R. By Proposition 5.9, there is a rectifiable such T. One cannot say there is an integral T since there is no control over the rectifiability of the boundary of T. It is convenient to think of a solution as a triple (Q, R, T), or to refer to R or T as solutions.

The minimal film may be thought of as the minimal cut separating the exterior sheet of Y (with the reference surface S) from the others. In particular, T may be viewed as prescribing a set of cuts and the gluing across them. These cuts are not a complete set of cuts for the construction of all of Y; they are sufficient only to separate the exterior sheet from the others. Conversely, any such set of cuts defines a feasible film, so the minimal film may be described as the minimal film-mass cut which separates the exterior sheet of Y from the others. The partial trefoil film (example 7.4, figures 1F and 7) is a case where the film is not a complete set of cuts.

The statement of (IMIN) explicitly requires that Q always have multiplicity $+1$ with respect to Y. This is to guarantee Q is a single cover of $[\![M]\!]$. I do not know if this explicit requirement is necessary, or whether the integrality and minimality are enough to rule out negative multiplicity sheets.

§**6.2. Real soap film problem.** For calibration methods to apply, a film problem must be stated in terms of real flat chains, and one cannot include the restriction that the exterior Q be integral. Hence the real minimization problem is:

$$
\begin{array}{ll}
\text{For} & Q \in \mathbf{F}_{m+1}(Y) \\
\text{minimize} & \Psi = \mathbf{M}^{\dagger}(\partial Q + S) \\
\text{such that} & p_{\#}Q = [\![M]\!].
\end{array}
\qquad \text{(RMIN)}
$$

Any solution Q has associated with it the flat m-chain $R = \partial Q + S$, representing the sides of the film in Y, and possibly many minimal film-mass representatives T of R. Since $\partial R = 0$, R is a normal current. By Proposition 5.8, T is a flat chain,

although one cannot say it is normal since there is no control over the mass of ∂T. As in the integral case, we will also refer to R and T as solutions of the problem.

To encourage integer density, one can take an integral reference current S, but it is unknown when this guarantees R is integral. Section 10.2 has some partial results. Nonintegral films do not interpret well as cuts in Y. Section 10.2 has more to say on how they might be interpreted.

§6.3. Adding triple line energy.

Triple lines may contribute to the energy of a surface [MT]. It is possible to accommodate this in the covering space model. To the basic sequence of spaces, the first singularity space Z_3 is added, and the mass of the triple line is included in the objective function. One can state both integral and real versions. The real version is stated here.

$$
\begin{aligned}
&\text{For} && Q \in \mathbf{F}_{m+1}(Y) \text{ and } T \in \mathbf{F}_m(W) \\
&\text{minimize} && \Psi = \alpha_1 \mathbf{M}(T) + \alpha_2 \mathbf{M}^\dagger(\partial T) \\
&\text{such that} && p_\# Q = [\![M]\!] \\
&&& q_\# T - \partial Q = S.
\end{aligned} \tag{TMIN}
$$

It is important for the compactness theorems that the mass of ∂T be controlled. The integral version would add the rectifiability and projection constraints on Q.

§6.4. General problem.

More generally, consulting diagrams 5.1 and 5.2, one could attribute energy to each order of homological singularity:

Minimize $\Psi = \alpha_2 \mathbf{M}(T) + \alpha_3 \mathbf{M}(P_3) + \ldots + \alpha_i \mathbf{M}(P_i) + \alpha_{i+1} \mathbf{M}^\dagger(\partial P_i)$

such that

$$
\begin{aligned}
p_\# Q &= [\![M]\!] \\
q_\# T - \partial Q &= S \\
r_{2\#} P_3 - \partial T &= 0 \\
&\vdots \\
r_{i-1\#} P_i - \partial P_{i-1} &= 0
\end{aligned}
$$

For the compactness theorems to work, it is necessary that all boundary masses be under control, hence none of the weights α_j can be zero.

One can compactly write the general problem by defining the film-mass of a film chain complex $\mathcal{P}$ as

$$
\mathbf{M}^\dagger(\mathcal{P}) = \alpha_0 \mathbf{M}(T) + \alpha_1 \mathbf{M}(P_1) + \ldots + \alpha_i \mathbf{M}(P_i)
$$

and letting $\mathcal{S}$ be the film chain complex representing the right sides of the system above. Then we can state the general problem as

$$
\begin{aligned}
&\text{Minimize} && \Psi = \mathbf{M}^\dagger(\mathcal{P}) \\
&\text{such that} && r_\# \mathcal{P} - \partial \mathcal{P} = \mathcal{S}
\end{aligned} \tag{GMIN}
$$

and possibly other constraints.

§6.5. Existence by compactness.

The direct method for proving the existence of absolute minima is to show that the set of feasible solutions is nonempty and compact and that the objective function is lower semicontinuous on the feasible set.

Theorem 6.1(Existence). *In the general minimization problem (GMIN), suppose Y is flat-m-bounded, each Z_i is flat-$(m + 2 - i)$-compact, and any auxiliary conditions define closed sets of chains. If there is a feasible solution with finite objective function, then there is an absolute minimum.*

Proof. Let Ψ_0 be the objective function value of a feasible solution. Since all $\alpha_i > 0$ and film-mass is bounded with respect to mass, we have some upper bound C on all $\mathbf{M}(P_i)$ and $\mathbf{M}(\partial P_i)$, depending on Ψ_0. Define

$$G_i = \{P \in \mathbf{F}_{m-i+2}(Z_i) : \mathbf{N}(P) \le 2C\}.$$

Then by Proposition 3.4(3),

$$G = G_1 \times \ldots \times G_j$$

is compact. The constraints define a closed subset of G, so we have a compact set F of feasible film chain complexes. Since mass (and hence Ψ) is lower semicontinuous and F is compact, there must be an absolutely minimal film chain complex. $\square$

Corollary 6.2. *Under the assumptions of Theorem 6.1, the problems (IMIN), (RMIN), and (TMIN) have solutions.* $\square$

7. EXAMPLES

This section will show how the covering space model can be applied to a wide variety of soap film examples. The reader should keep in mind that while this section talks about this and that soap film, only in the simplest cases is the film known exactly.

All the examples in this section are of soap films on wire frames in three dimensions. The case of a soap film on a wire frame is done with a certain choice of ambient manifold M, primary covering space Y, and pair covering space W. Intuitively and physically, a soap film on a frame is made by immersing the frame in a soap solution and then withdrawing the frame so the film shrinks over the frame. Let the frame be a bounded closed set B in $\mathbb{R}^3$, and let E be a closed bounded region containing B. Let $M = E - B$. E corresponds to the initial blob of soap solution. Some care must be used in choosing E, since the boundary of E will act as an obstacle to the film. Generally, one wants to be sure the film stays away from the boundary of E, and one can ensure that by having the boundary of E have positive mean curvature.

A covering space Y can be made by taking several copies of M and cutting and gluing them together. The cuts are two-dimensional surfaces. We will require that Y have a sheet to which the boundary of E can be lifted to form the reference current S, which is the outer boundary of the exterior chain Q. This sheet will be referred to as sheet A. Each example will indicate how the cutting and gluing is done.

Unless otherwise specified, the pair covering space will be the full pair covering space.

Most of these examples are included in the Polycut program mentioned in section 1.

Example 7.1. Disk. Take B to be a unit circle (circumference, not disk). Take Y to be the double cover of $M = E - B$. One may construct Y by taking two copies of M (sheets A and B), making a cut surface on the disk spanning the circle, and gluing sheets A and B to each other across the cut. The pair covering space W has two sheets, AB and BA, and is connected. Take the exterior Q to be just the current of A. The film then has two sides, one on each sheet. The same film solves the integral and real problems. The fundamental group of M is just $\mathbb{Z}$, so the possible connected covering spaces Y are distinguished only by the number of sheets. The same film results regardless of the number of sheets.
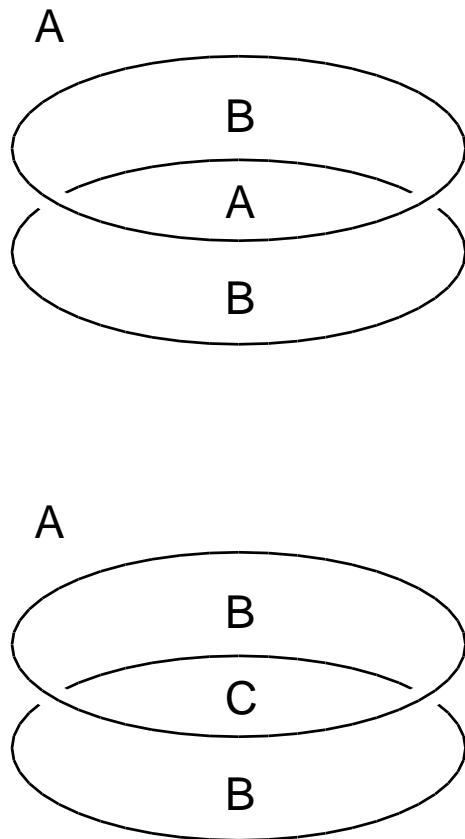

Figure 4. Soap films on two rings. Top row uses a double cover for Y. Top left shows background sheets as seen from sheet A. Corresponding film at top right is a catenoid. Bottom row uses a cyclic triple cover for Y. Bottom left shows background sheets as seen from sheet A. Note that going through the center gets to a different sheet. Thus the film has disk in middle, as shown at bottom right. (Example 7.2)

Example 7.2. Catenoid. The boundary set B will be two parallel coaxial unit radius rings. If the rings are too far apart, the only soap film is two parallel disks. If they are closer together, one can get different soap films, depending on which covering space is used. If the double cover is used, the solution is the catenoid. The top row of figure 4 illustrates this. There the covering space is shown by indicating which sheet you would see in the background. If one looks so that the line of sight

passes through only one ring, then one sees a different sheet in the background. Hence the minimal film must block all such lines of sight. Since one does not see a different sheet when looking through both rings, there does not need to be film across the center. The catenoid has these properties, and it can be proved to be the solution using the techniques of chapter 8.

A triple cover may be constructed by making cuts of the two disks and gluing three sheets cyclicly the same way across the two cuts, so that if you looked from sheet A through both circles you would see sheet C. Thus the minimal film has to block these lines of sight. The soap film for this covering is a catenoid with a disk across its middle, as shown in the bottom row of figure 4. If the disk cuts were glued in opposite cycles, then going from sheet A through both rings would wind up back on sheet A. Hence the minimal film would be the catenoid again.
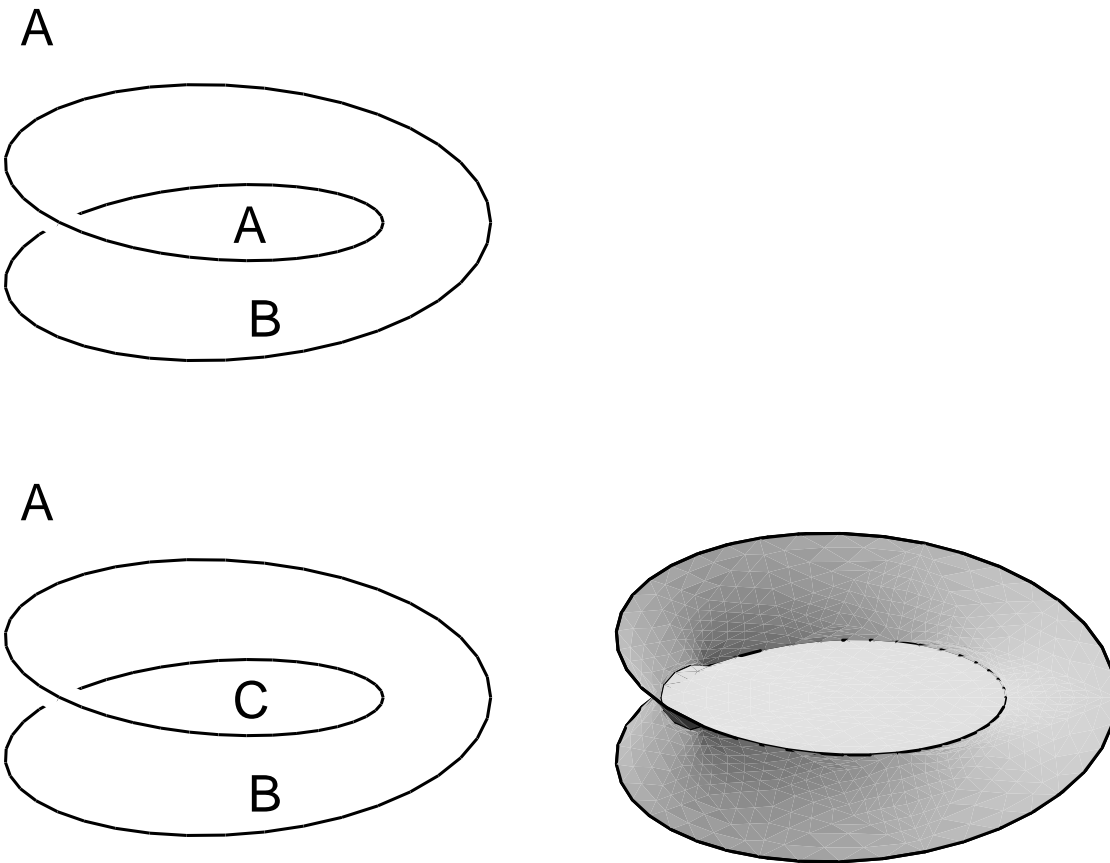

Figure 5. Soap films on a Mobius band boundary. Top row uses a double cover for Y. Top left shows background sheets as seen from sheet A. The corresponding film at top right is a Mobius band. Bottom row uses a triple cover for Y. Bottom left shows background sheets as seen from sheet A. Note that going through the center gets to a different sheet. Thus the film has disk in middle, as shown at bottom right. (Example 7.3)

Example 7.3. Mobius band. The boundary B is an unknotted loop folded over on itself, shown in figure 5. Topologically, B and its complement M are the same as in the disk example, however the films turn out differently. The double cover

leads to a Mobius band soap film (top of figure 5). Note this film is unorientable and hence cannot be handled with many previous soap film models. But as the boundary of sheet A, the two sides of the film form an orientable chain.

The triple cover leads to a Mobius band with a central film (bottom of figure 5), because going through the middle gets to a different sheet and hence there must be a film on the way. Note the film has a short section of triple junction on the left.

Example 7.4. Trefoil knot. The boundary B is a 2,3 torus knot (figure 6). The boundary itself is topologically a circle, but the complement is topologically different from the complements in examples 7.1 and 7.3, having a nontrivial fundamental group. If Y is the double cover, then the film is a Mobius band with three half-twists (top of figure 6). If Y is the triple cover with all three sheets cyclically connected around B, then the film is the complete film on the knot with a central film and three side lobes joined to it along triple lines (bottom of figure 6).

If Y is a properly chosen triple cover, the film is supported by only part of B. This Y is diagrammed in figure 7. Start with three copies of M, called sheets A, B, and C. Make a cone-shaped cut from the center out to B on each sheet. The dashed lines in the top left diagram of figure 7 show where the cuts self-intersect. These intersection lines divide the cut surface into three lobes. On the top lobe, glue A to A and B to C. On the lower right lobe, glue B to B and A to C. On the third, glue C to C and A to B. If one is in sheet A and looks through the knot, one sees the background sheets as shown at the top right. There are two branch curves in this covering space (actually, in $\bar{Y}$). One has branch order 2 and the other has branch order 1. Both are complete trefoils covering the original trefoil in the base space. The order 1 branch curve is a removable singularity of Y; it can be sewn into Y to yield a manifold. In terms of how it affects the background view, it is an "invisible wire". So that branch curve is irrelevant to the soap film. The film will be inside the convex hull of the order 2 branch curve as seen from A. Hence the film doesn't touch the whole knot. Parks [PH] and Drachman & White [DW] have studied (by very different methods) the circumstances under which a soap film can touch only part of a boundary curve. They show that knotted curves always support a film that does not completely touch the curve. A third proof of this fact using covering spaces is given below in section 11.3.
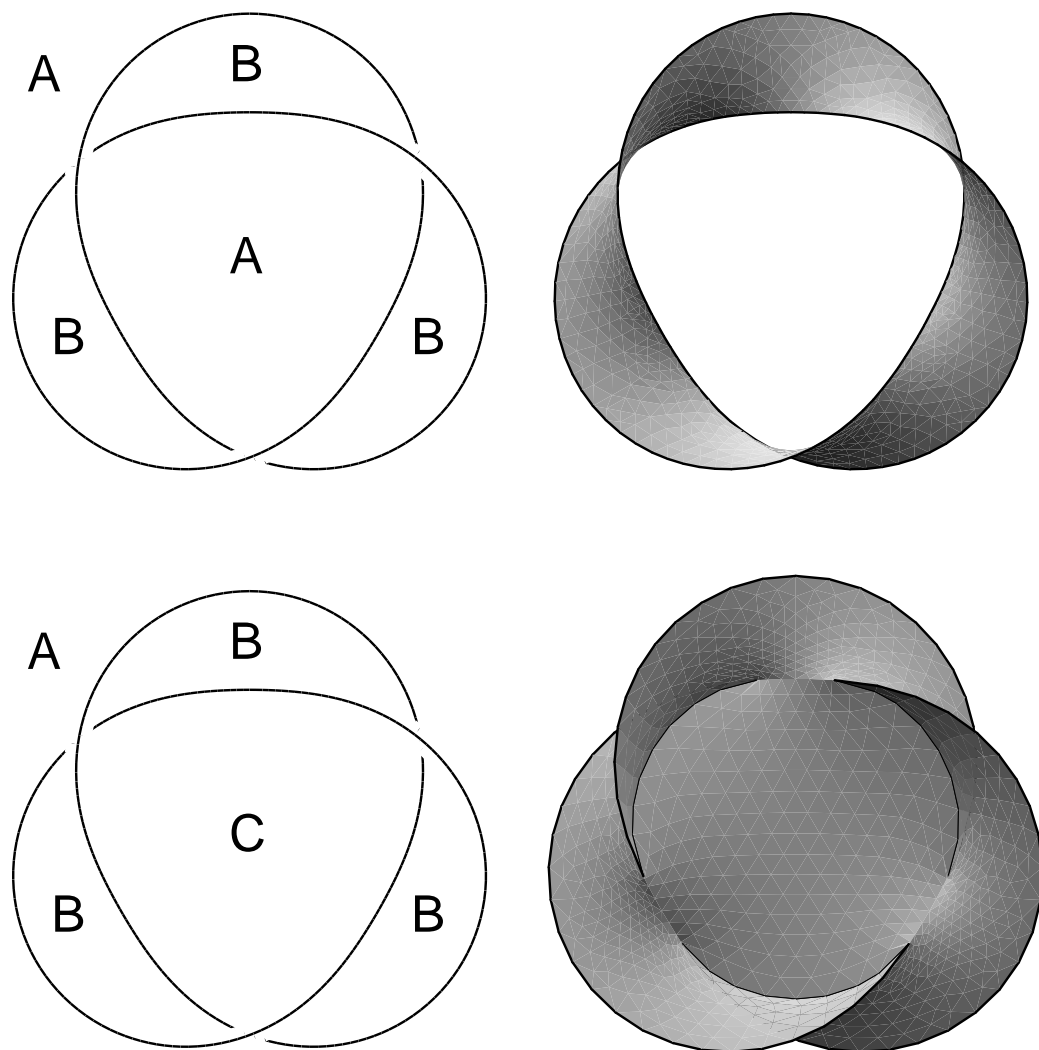
Figure 6. Soap films on a trefoil knot. Top row uses a double cover for Y. Top left shows background sheets as seen from sheet A. Corresponding film at top right is a Mobius band with three half twists. Bottom row uses a cyclic triple cover for Y. Bottom left shows background sheets as seen from sheet A. Note that going through the center gets to a different sheet. Thus film has disk in middle, shown at bottom right. (Example 7.4)
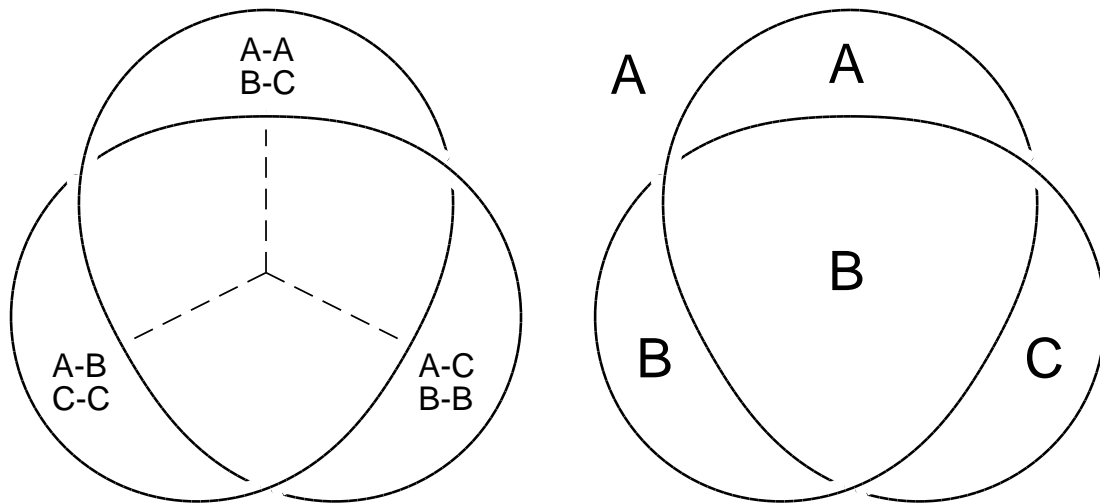
Figure 7. A soap film partially touching a trefoil knot. Top left picture shows cuts and gluings for the triple cover. The cut surface is a cone from the center to the knot. The dashed lines are the cut self-intersections. Each cut lobe shows the gluings; e.g. the top lobe has sheet A glued to A and sheet B glued to sheet C both ways. The upper right shows the background sheets as seen from sheet A. The lower picture shows the soap film. Note that the bare wire corresponds to a branch curve of order 1 in Y. (Example 7.4)

There is a fourth surface bounded by the trefoil knot that is a smooth orientable manifold. It is shown in figure 8. In knot theory, it is known as the Seifert surface of the knot. One can use a four-fold covering space Y that is cyclicly connected around the boundary. The key is to restrict the pair space W to contain only successive pairs. That is, if Y consists of sheets A, B, C, and D, then W consists of sheets AB, BC, CD, and DA. If a single central film were to form, it would be pair AC or BD. But an AC pair film has to be decomposed as AB + BC to calculate its film-mass, so an AC film has twice the film-mass of an AB film. The result is that two layers form in the center in figure 8. The general problem of an oriented smooth manifold bounded by a given curve is considered in section 12.2 below.
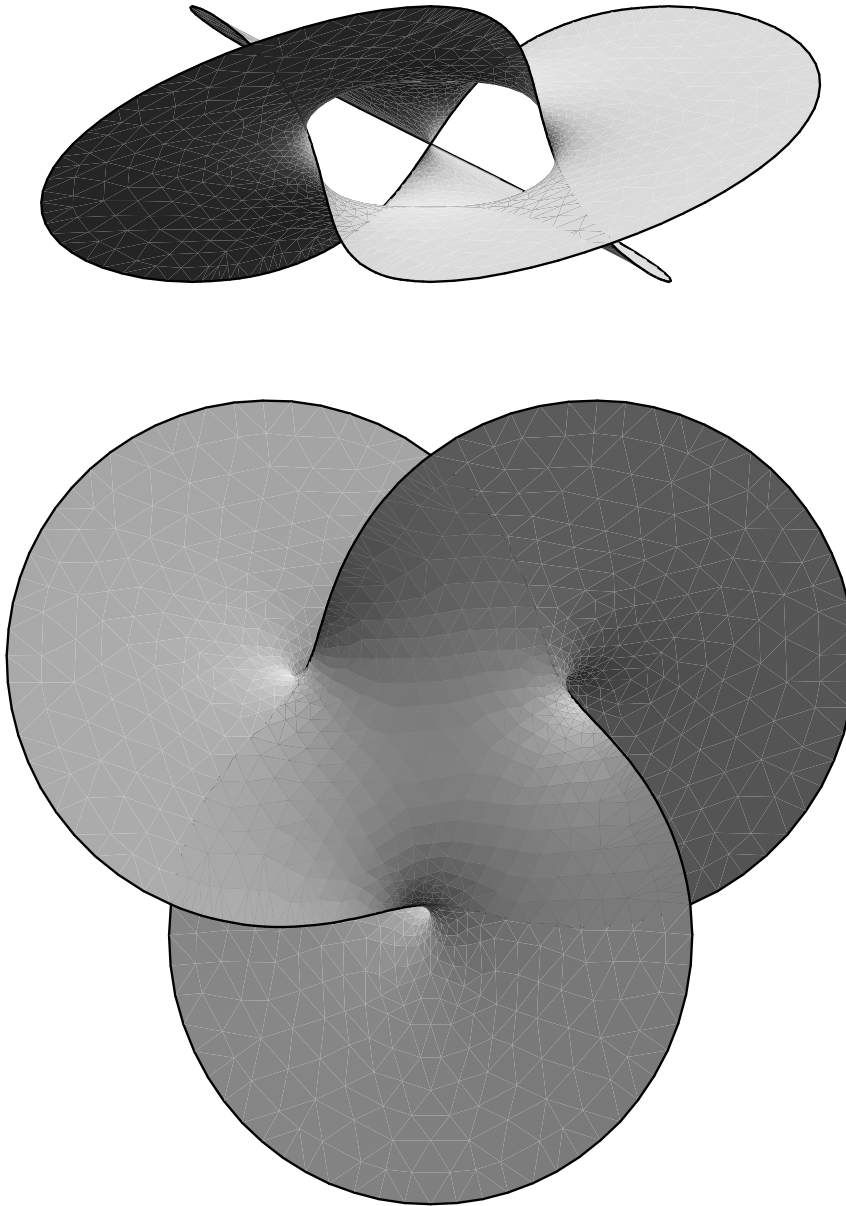
Figure 8. An oriented manifold bounded by a trefoil knot. (Example 7.4)

Example 7.5.   Penetrating loop. This is an example where the standard integral current techniques fail. It was suggested to the author by Frank Morgan. The boundary consists of two unit circles (see figure 9), one vertical in the y-z plane and centered at the origin, and the other horizontal in the x-y plane and centered at $(0, -H, 0)$. Start with $H > 1$ and a disk soap film spanning the vertical circle. As H goes slightly below 1, the horizontal circle pushes forward on the film. As a current in E, the film has no boundary on the horizontal circle and hence is free to pull away from it back to a flat disk. But with the proper covering space, we can get the desired film. Take three sheets, A, B, and C, and make a disk cut spanning the horizontal circle and a hemisphere cut spanning the vertical circle (so the cuts

don't intersect, for simplicity). On the horizontal cut, glue A to A and B to C. On the hemisphere cut, glue A to B and C to C. Then as long as $H < 1$, the calibration techiniques of chapter 8 can be used to show the disk is not the minimal film. We can construct a flow with flux greater than π. This flow can be made of two parallel beams at small angles θ, $-\theta$ to the $x - y$ plane, coming from above and below. The flux is

$$\pi \cos(\theta) + 2\alpha \sin(\theta)$$

where α is the area of the vertical loop in front of $x = 0$. Both circles give rise to two branch curves, one of order 2 and one of order 1. The rear of the horizontal circle turns out to be an invisible wire.
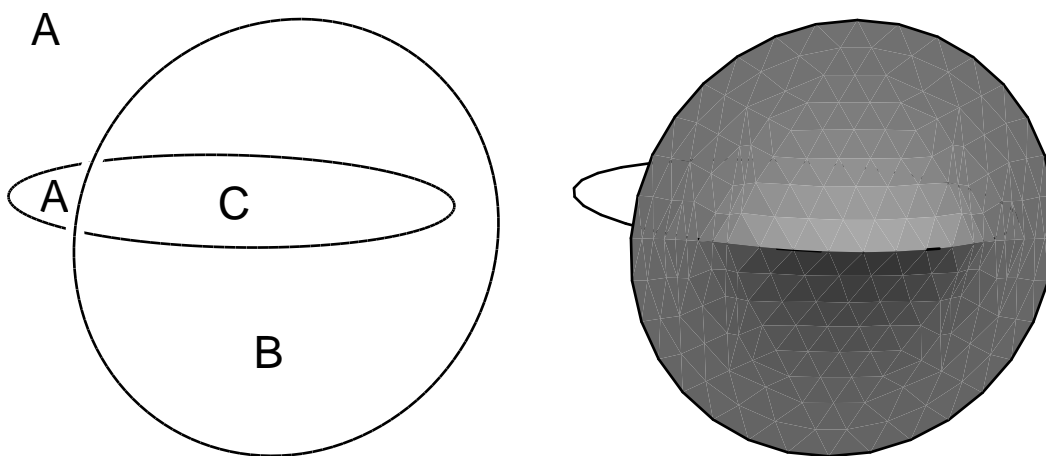

Figure 9. Soap film on a loop penetrated by another loop. The left shows the background sheets as seen from sheet A. The corresponding film at the right shows how the second loop is an obstacle to the film, even though it has no boundary on it as a simple integral current. (Example 7.5)

Example 7.6. Double catenoid. Consider the wire boundary shown in figure 10. It is essentially two sets of catenoid rings joined together at right angles. The boundary is topologically an unknotted circle. The film that corresponds to the universal cover (or any nontrivial cover) of the complement of the boundary consists of two joined catenoids, as shown in the lower part of figure 10. One can check that this surface does intersect all closed paths that link the boundary. Also note that any path going through the center of both rings of a pair can be homotoped outside the boundary by going between the other pair of rings. Therefore, in any covering space, the background sheet seen through the center of a pair of rings will be sheet A. Hence no film spans the centers of the rings. This is an example of a film that does not have a simply connected complement even though the universal covering space is used.
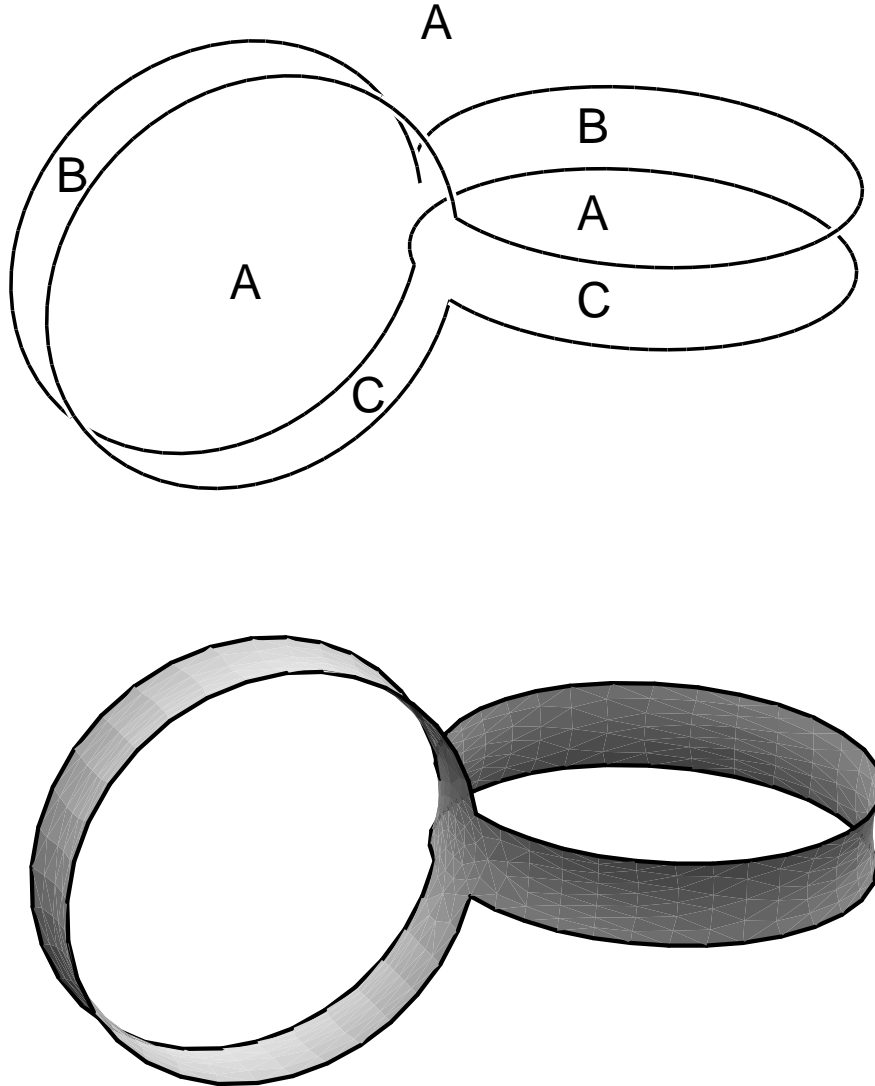
Figure 10. Two catenoids joined at right angles. The top image shows the boundary wire, which is topologically an unknotted circle. The bottom picture shows the film that forms using any cover of the complement of the boundary. Note that this film does not have a simply connected complement. (Example 7.6)

Example 7.7. Double catenoid with one disk. With the wire boundary of the previous example, it is physically possible to get a film that is a double catenoid with a disk spanning one of the catenoids, as shown in figure 11. This film cannot be produced with a covering space of the entire complement of the boundary, from the argument of the previous example, but it can be produced if one adds an invisible wire as an obstacle. In figure 11, this is the line L perpendicular through the center of the left pair of rings. This no longer permits paths through the right disk to homotope to the outside. An appropriate three-sheeted covering space may be constructed by cutting and gluing three copies of M as follows. One cut is the same as the film of figure 10 (a double-catenoid cut). It glues sheets A, B, and C

together cyclically. The other cut is a half-plane extending horizontally to the right of line L and in between the right pair of rings. Outside the double-catenoid cut, It glues sheet A to itself, and sheets B and C reciprocally to each other. Inside, it glues sheet C to itself, and sheets A and B reciprocally. Note that none of these cuts disrupt the reference surface S on the edge sheet A. In particular, the line L is invisible in sheet A and does not touch the film. A path through the right pair of rings arrives on sheet B, and the disk spanning the right catenoid is the bottleneck for such paths. If one inserted this film in the covering space of the previous example (without invisible wires), then the two sides of the disk would be on the same sheet and cancel out.
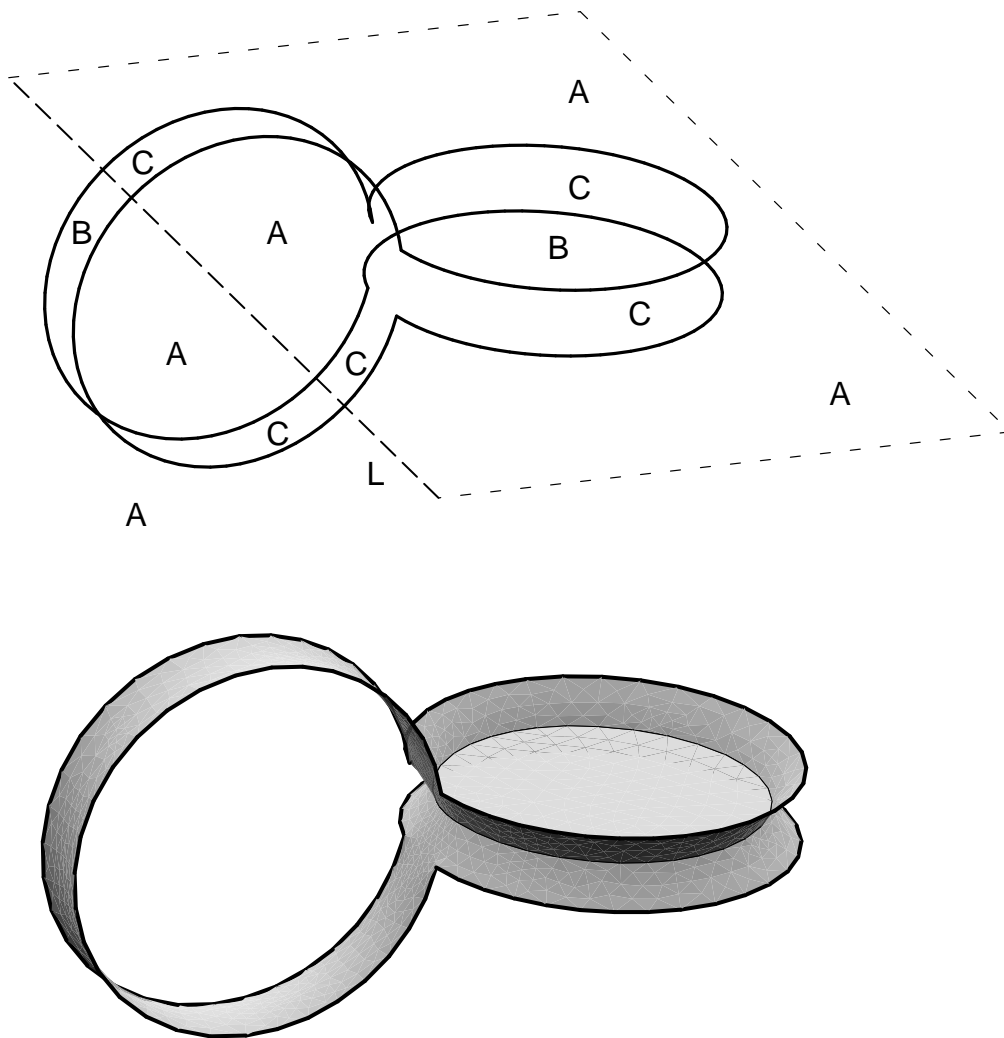

Figure 11. Two catenoids joined at right angles with a disks in one. The top picture shows the construction of the covering space. The heavy dotted line is a boundary that is invisible on sheet A. A half-plane cut extends to its right. The other cut surface is the same shape as the double catenoid surface of figure 10. Labelling shows the background sheets seen. The lower picture is the corresponding film. (Example 7.7)

Although line L is invisible, its presence would be felt if the left pair of rings were spread apart so the catenoid strip became unstable and tried to collapse into a pair of disks on the left and a catenoid strip on the right. Figure 12 shows the effect of wire L for a double catenoid boundary in which the left pair of rings has been shrunk and moved apart for convenience. The top of figure 12 shows the film resulting from using the covering space of example 7.6. Moving the left rings apart causes the catenoid to collapse to a pair of disks and a strip. Physically, the film of figure 11 would collapse to the same shape if its left rings were spread. But if wire L is present, then the collapsing film finds L to be an obstacle, and the film at the bottom of figure 12 results. This film results from the covering space described earlier in this example.
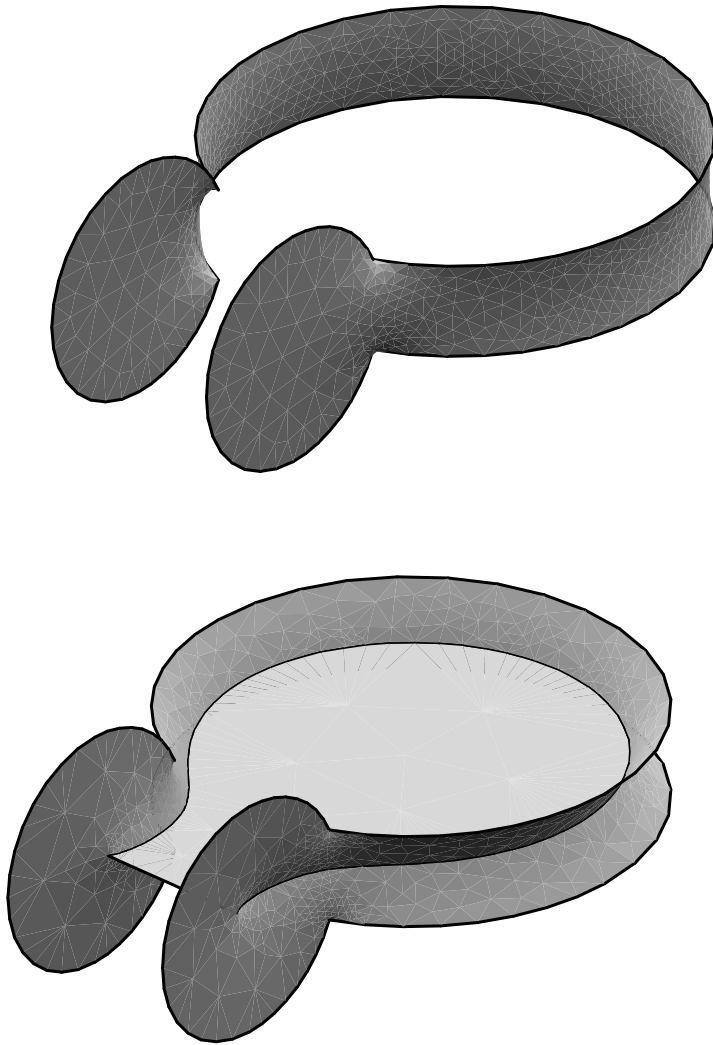

Figure 12. Effect of line L. The top shows a deformation possible for the film of figure 11 which would be possible if line L were missing. However, if line L is present, then the deformation is as shown at the bottom. For convenience, the left rings have been shrunk and moved farther apart. (Example 7.7)

If one puts a half-twist in the strip of catenoid on the left of figure 11, then the ordinary triple cover generates the film with no need for invisible boundary wires.

Example 7.8. Double catenoid with two disks. An example using more invisible wires is the double catenoid with disks in both catenoids, shown in the bottom of figure 13. Add boundary wires L_1 and L_2 through the centers of the left and right pairs of rings, respectively. The covering space Y has four sheets, and three cut surfaces are needed. The first two cuts are the same as those in example 7.7, with the addition of gluing sheet D to itself. A new cut is made with a half-plane H_2 to the left of line L_2 which passes between the left pair of rings. The outer part of H_2 glues sheets A and B to themselves, and C and D to each other reciprocally. The inner part glues sheets A and D reciprocally, and sheets B and C to themselves. The top of Figure 13 shows the background sheets seen from sheet A. Note that some lines of sight go through as many as four cut surfaces. Due to the way the gluing is done, line L_1 has no effect on sheets A and D, and line L_2 has no effect on sheets A and B. The disk in the left catenoid lies between sheets A and D, so line L_1 doesn't disrupt it. Likewise, the disk in the right catenoid lies between sheets A and B, and isn't disrupted by line L_2. This corresponds to the physical fact that one can put a wire through a soap film without affecting its shape (except for a small effect due to the wire's nonzero thickness).

The general lesson of these catenoid examples is that stable but not absolutely minimal films may be made absolutely minimizing by adding "invisible wires" that act as obstacles to deformations but don't otherwise affect the shape of the film. In effect, invisible wires can convert a local minimum into a global minimum by modifying the problem elsewhere.

An example of a soap film that deformation retracts to the empty film was found by J. F. Adams [RE, Appendix]. It is pictured in figure 14. It consists of two catenoids-with-disks with two slightly twisted triple bridges between them. I leave finding the appropriate invisible wires and covering space as an exercise for the reader.
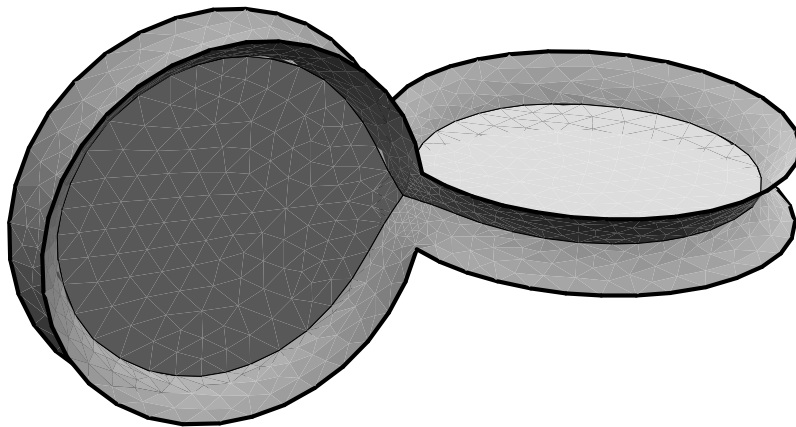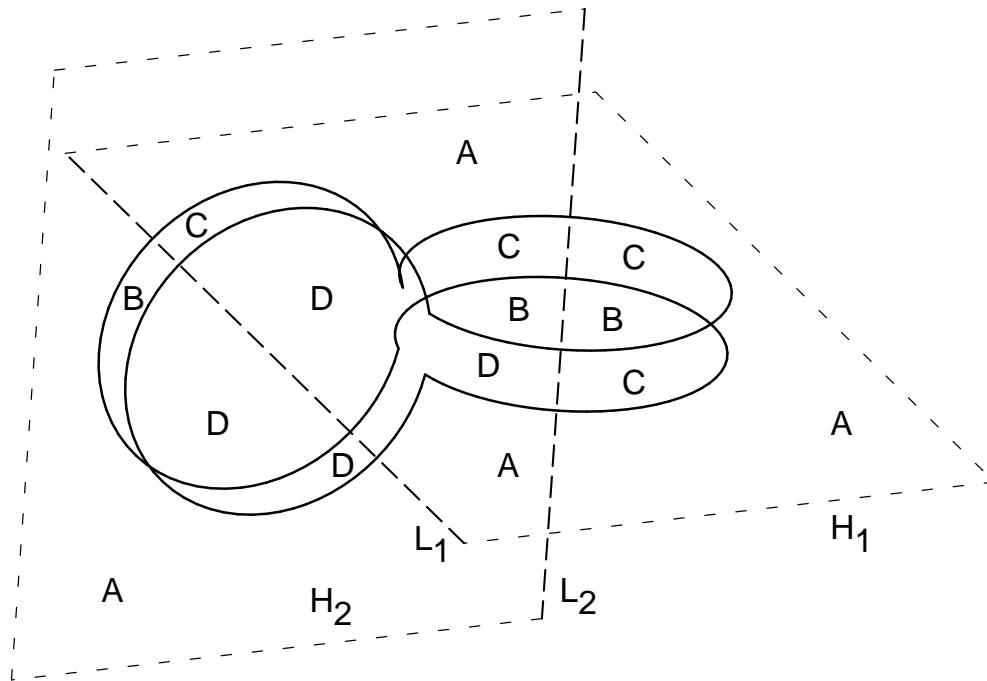
Figure 13. Two catenoids joined at right angles with disks in both. The top picture shows the construction of the covering space. The heavy dotted lines are boundaries that are invisible on sheet A. Half-plane cuts extend to the right of the horizontal line and the left of the vertical line. The other cut surface is the same shape as the double catenoid surface of figure 10. Labelling shows the background sheets. The lower picture is the corresponding film. (Example 7.8)
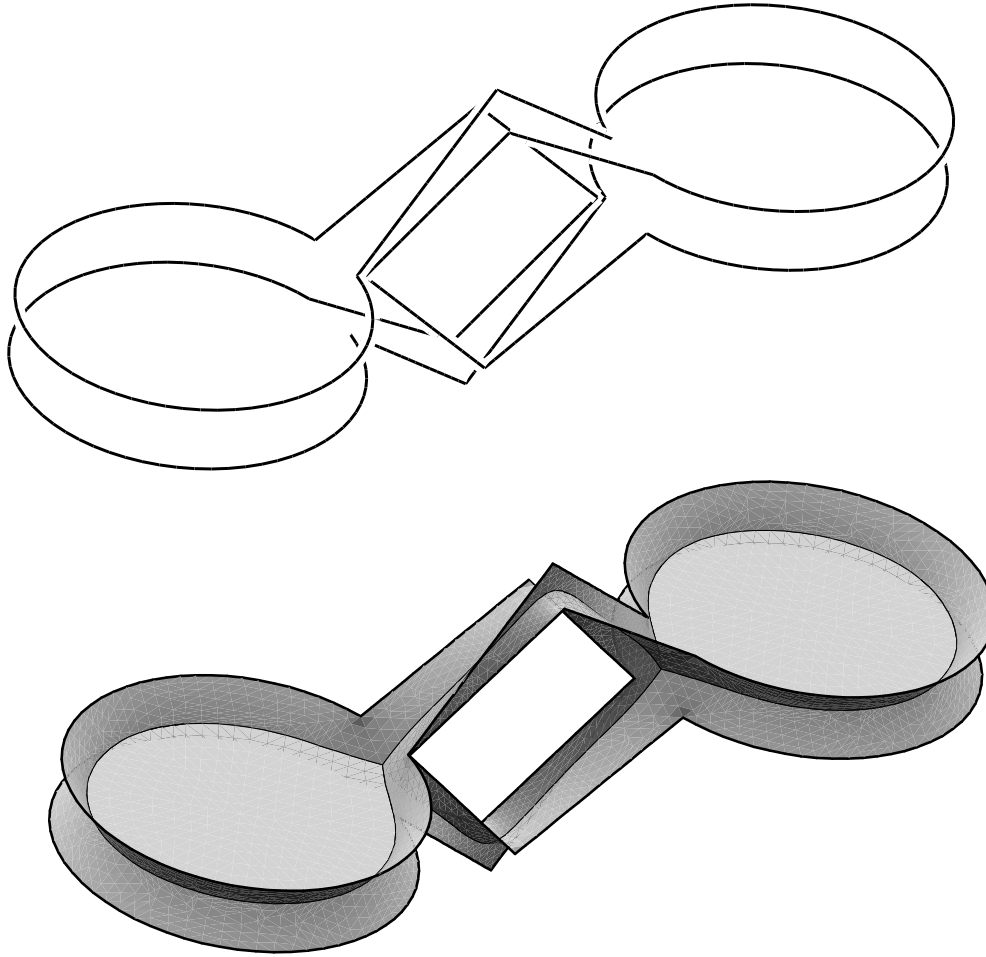
Figure 14. A soap film that is a deformation retract to its boundary. (Example 7.8)

Example 7.9. Flat spiral. This is an example of an unknotted simple closed curve for which every covering space gives a different film, and there are uncountably many films altogether when invisible wires are used. See figure 15. The boundary wire is a logarithmic spiral with a radial edge from the center of the spiral slanting above the spiral, and then joining the outer point of the spiral. The covering spaces of the complement are the N-fold covers. For each N, the film fills in the circular strips in a pattern of $N-1$ filled strips and one empty strip. The ends of the strips join in various ways to a radial strip descending from the radial slant edge. With invisible wires, one can arrange that any subset of the lanes have film, giving an uncountable number of different films.

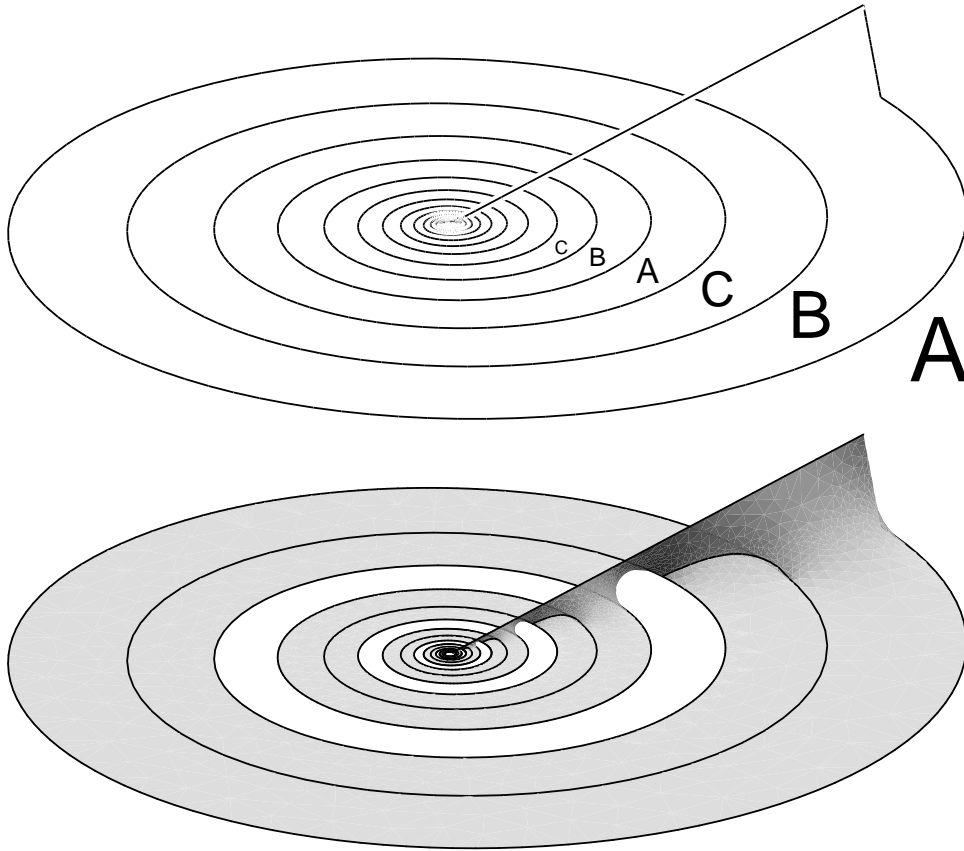Figure 15. Simple closed curve bounding a different film for each different number of sheets in the covering space. The top picture show the background sheets for the triple cover. The bottom picture shows the corresponding film. The triple line where lane B ends and lane C begins is tangent to the boundary at both of its ends. With invisible wires, it bounds uncountably many. (Example 7.9)

Example 7.10. Crossed tripods. It may happen that the real minimizer has smaller mass than the integral minimizer. Take Y to be two copies of the tripod covering space of figure 2, rotated 60° with respect to each other, as shown in the lower part of figure 16. Take S to be both tripod S's, that is, the boundaries of sheets A and D. This violates previous assumptions that S is on just one sheet, so this is not an ordinary soap film problem. The integral minimal soap film is presumably two tripods for mass 6 (top left of figure 16). But there is a real solution of less mass (top right of figure 16). Even though I cannot prove the crossed tripods is the integral minimum, this example strongly suggests that having an integral S does not guarantee the real minimum is integral.
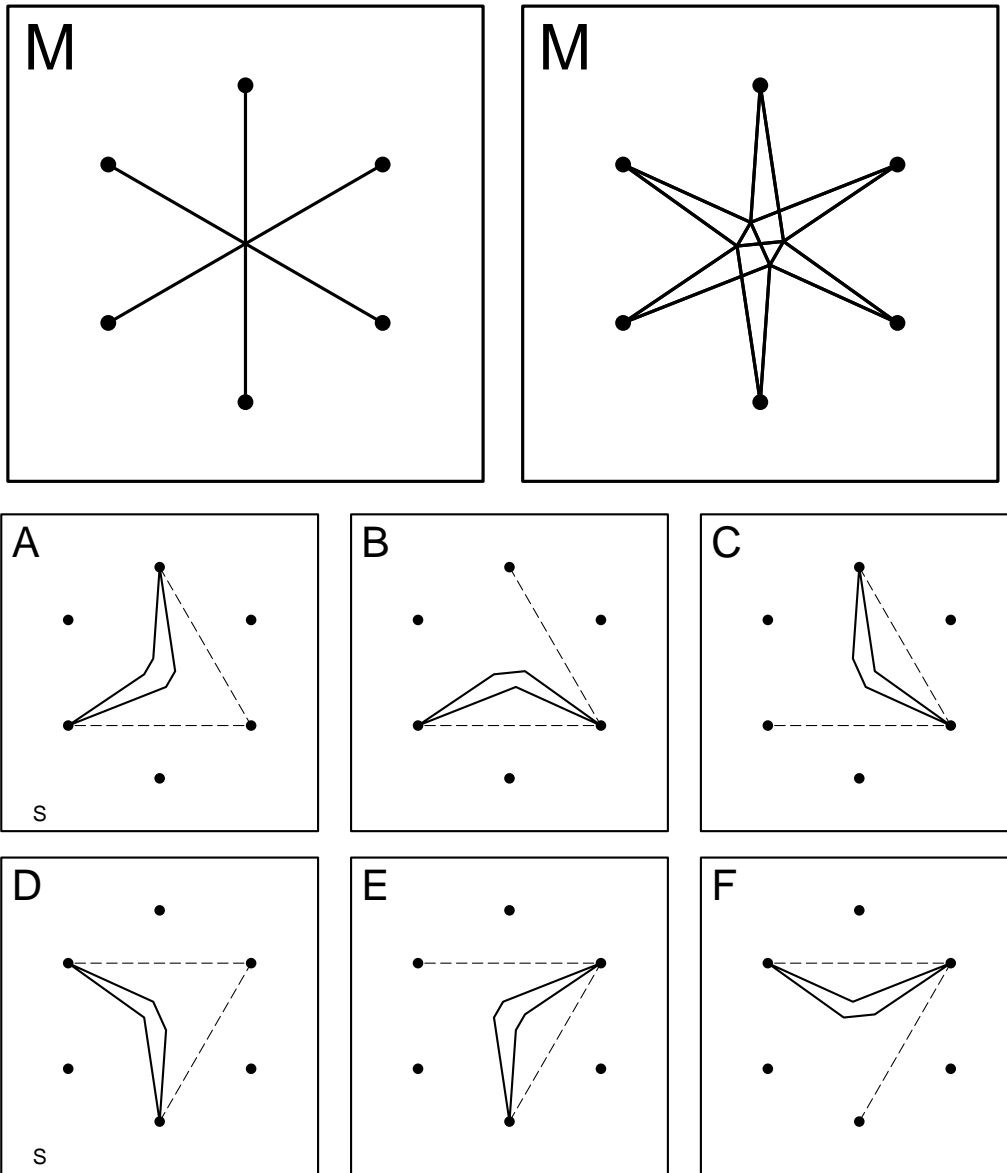
Figure 16. Real current beating crossed integral tripods. Top left shows the presumed integral minimum with mass 6. Top right shows a real solution where all lines have density 0.5 and the mass is 5.9638. Second and third rows show the six sheets of the covering space. (Example 7.10)

8. Proof of minimality: calibration

§8.1. Calibration. Suppose V is a vectorspace with norm $||\cdot||$ and V^* is its dual space with dual norm $||\cdot||^*$. Then for any $v \in V$ and $\omega \in V^*$,

$$\langle \omega, v \rangle \leq ||\omega||^* ||v||.$$

If equality holds, we shall say that ω and v *calibrate* each other. Given a nonzero $v \in V$, the Hahn-Banach Theorem implies that there is always a calibrating covector $v^* \in V^*$ with $||v^*|| = 1$. The converse may not be true if V is not reflexive; $\omega \in V^*$ may not have a calibrating vector ω^*.

Calibration is a general method for proving that a particular current is absolutely minimizing in its homology class. Suppose T is the purported minimizing current. A calibrating form ω for T has $d\omega = 0$, $\mathbf{M}(\omega) = 1$, and $\langle T, \omega \rangle = \mathbf{M}(T)$. If R is another current homologous to T, then

$$\mathbf{M}(T) = \langle T, \omega \rangle = \langle R, \omega \rangle \leq \mathbf{M}(R).$$

Federer [F1,5.4.19] used a certain differential form to prove that complex varieties are absolutely area minimizing as currents. He also developed a calibration theory (although not under that name) for real flat chains in [F2]. The model in this paper is an extension of that theory. A general theory of calibrations for manifolds was presented in [HL].

A key advance came with the paired calibration technique of [LM] and [B1]. Here a film is represented as the dividing surface between several immiscible fluids. Each fluid is represented as an $(m + 1)$-chain $Q_i \in \mathbf{F}_{m+1}(\mathbb{R}^{m+1})$. Each fluid has a fixed outer boundary S_i (an m-chain), and has a flow (closed m-form) ω_i associated with it. Further, the differences between the flows satisfy

$$\|\omega_i(x) - \omega_j(x)\| \leq \alpha_{ij} \text{ for all } x$$

where α_{ij} is the surface tension between fluids i and j. The boundary between fluid i and fluid j is the m-current T_{ij}, with $T_{ji} = -T_{ij}$, and the boundary of fluid i is $S_i + T_i$, where $T_i = \sum_j T_{ij}$. If one has flows for which the total flux through the interfaces is equal to the total energy of the interfaces, then one has a calibration and a proof that T is minimal. For if R is a competing interface, then

$$
\begin{aligned}
energy(R) &= \sum_{i<j} \alpha_{ij} \mathbf{M}(R_{ij}) \\
&\geq \sum_{i<j} \langle R_{ij}, \omega_i - \omega_j \rangle \\
&= \sum_i \langle R_i, \omega_i \rangle \\
&= \sum_i \langle T_i, \omega_i \rangle \\
&= energy(T).
\end{aligned}
$$

[LM] used this to show that the cone over the $(n - 2)$-dimensional skeleton of an n-simplex is minimizing in any dimension $n \geq 2$, and [B1] showed that the cone over the $(n - 2)$-dimensional skeleton of an n-dimensional hypercube is minimizing in any dimension $n \geq 4$. The original method cannot handle unoriented surfaces like the Mobius band or the trefoil knot film since it requires the film to bound distinct regions. The covering space model proposed here removes those limitations.

Now we describe the spaces of flows used in the covering space model. A *smooth flow* is a k-form $\omega \in \mathcal{D}^k(Y)$ such that $d\omega = 0$. The space of smooth flows is not broad enough to provide calibrating flows. A *flat flow* in Y will be a flat cochain $\omega \in \mathbf{F}^k(Y)$ such that $d\omega = 0$. Note that either type of flow can have source or sink

only across the closed boundary of Y, and there is no leakage of flux out of the open boundary. The set of flat flows on Y will be denoted $\mathcal{W}(Y)$.

Our soap films are flat chains, so flat cochains are valid flows on them. Flat cochains are very convenient to use. A flow is designed to have zero divergence and bounded magnitude, and any flow one can actually construct is Lebesgue measurable, so all flows are automatically flat cochains and hence integrable on soap films.

Finding a calibrating flow is the dual problem to finding a minimal chain. It is essentially the same duality found in linear programming and network max-flow, min-cut theory. In fact, the easiest way to formulate the calibration problem is to write down the chain minimization problem as a linear programming problem, and take the transpose to find the dual problem. Recall that the basic minimization problem (GMIN) is:

$$\text{Minimize } \Psi = \alpha_2 \mathbf{M}(T) + \alpha_1 \mathbf{M}(P_3) + \ldots + \alpha_j \mathbf{M}(P_j)$$

such that

$$
\begin{aligned}
p_\# Q &= [[M]] \\
q_\# T - \partial Q &= S \\
r_{2\#} P_3 - \partial T &= 0 \\
&\vdots \\
r_{j-1\#} P_j - \partial P_{j-1} &= 0
\end{aligned}
$$

To formulate the dual maximum flow problem, first introduce one flat cochain variable of the proper dimension and domain for each constraint, i.e. $\omega_i \in \mathbf{F}^{m-i+1}(Z_i)$. The dual objective function consists of these cochains contracted with the right sides of the constraints. The dual problem has one constraint for each variable of the original. The coefficients are the transposed duals of the original coefficients (∂ becoming d), and the right hand side bounds come from the minimization objective function coefficients. Hence the dual problem is:

$$\text{Maximize } \Phi = \langle [[M]], \omega_0 \rangle + \langle S, \omega_1 \rangle$$

such that

$$
\begin{aligned}
p^\# \omega_0 - d\omega_1 &\leq 0 \\
q^\# \omega_1 - d\omega_2 &\leq \alpha_2 \\
r_2^\# \omega_2 - d\omega_3 &\leq \alpha_3 \\
&\vdots \\
r_{j-1}^\# \omega_{j-1} &\leq \alpha_j.
\end{aligned}
$$

These inequalities must be interpreted as being pointwise true for any unit k-vectors, so we write the problem as

$$\text{Maximize } \Phi = \langle [[M]], \omega_0 \rangle + \langle S, \omega_1 \rangle$$

such that

$$p^{\#}\omega_0 - d\omega_1 = 0$$
$$\mathbf{M}(q^{\#}\omega_1 - d\omega_2) \leq \alpha_2$$
$$\mathbf{M}(r_2^{\#}\omega_2 - d\omega_3) \leq \alpha_3$$
$$\vdots$$
$$\mathbf{M}(r_{j-1}^{\#}\omega_{j-1}) \leq \alpha_j.$$

Since ω_0 is a top-dimensional chain, we may write $\omega_0 = d\lambda$. If we replace ω_1 with $\omega_1 - p^{\#}\lambda$, then the problem becomes

$$\text{Maximize } \Phi = \langle S, \omega_1 \rangle$$

such that

$$d\omega_1 = 0$$
$$\mathbf{M}(q^{\#}\omega_1 - d\omega_2) \leq \alpha_2$$
$$\mathbf{M}(r_2^{\#}\omega_2 - d\omega_3) \leq \alpha_3 \qquad\qquad \text{(GMAX)}$$
$$\vdots$$
$$\mathbf{M}(r_{j-1}^{\#}\omega_{j-1}) \leq \alpha_j.$$

Henceforth (in this version of this paper) we will consider just the simple film problem, without singularity mass:

$$\text{Minimize } \Psi = \mathbf{M}(T)$$

such that

$$p_{\#}Q = [[M]]$$
$$q_{\#}T - \partial Q = S. \qquad\qquad \text{(RMIN)}$$

The dual basic maximal flow problem is (with $\alpha_2 = 1$)

$$\text{Maximize } \Phi = \langle S, \omega \rangle$$

such that

$$d\omega = 0$$
$$\mathbf{M}(q^{\#}\omega) \leq 1$$

or, in term of film-comass,

$$\text{Maximize } \Phi = \langle S, \omega \rangle$$

such that

$$d\omega = 0$$
$$\mathbf{M}^{\dagger}(\omega) \leq 1. \qquad\qquad \text{(RMAX)}$$

Theorem 8.1 (Calibration). *Suppose Q and ω are feasible solutions to the real minimization problem (RMIN) and maximization problem (RMAX) respectively, and $\mathbf{M}^{\dagger}(\partial Q + S) = \langle S, \omega \rangle$. Then Q and ω are optimal solutions. Furthermore, if $T \in \mathbf{F}_m(W)$ is a minimal film-mass representative of $\partial Q + S$, then $(q^{\#}\omega)(w)$ calibrates $\vec{T}(w)$ for $||T||$-almost all $w \in W$.*

Proof. If Q and ω are any feasible solutions and T is a minimal film-mass representative of $\partial Q + S$, then

$$
\begin{aligned}
\langle S, \omega \rangle &= \langle q_{\#}T - \partial Q, \omega \rangle \\
&= \langle T, q^{\#}\omega \rangle - \langle Q, d\omega \rangle \\
&\leq \mathbf{M}(T)\mathbf{M}(q^{\#}\omega) - 0 \\
&\leq \mathbf{M}(T) = \mathbf{M}^{\dagger}(\partial Q + S)
\end{aligned}
$$

Hence, if $\langle S, \omega \rangle = \mathbf{M}(T)$, then ω and Q must be optimal. The only way equality can hold is if the calibration is as stated. $\square$

Remarks. In the case that there are multiple minimal films, each optimal flow must calibrate all of them.

Note that the calibration criterion for minimality makes no assumptions on the compactness or boundedness of the covering spaces involved. Hence one may be able to prove certain films are absolutely minimizing in spaces where existence proofs do not apply.

Calibrations are not guaranteed for the solution of the integral problem (IMIN). Solutions to the integral problem are feasible for the real problem, so the integral solution may happen to be the real solution also. In this case, we can calibrate the integral solution.

§8.2. Maximal flows.   Maximal flows may be defined without reference to minimal films. For convenience, we will name the sets of flows of film-comass at most 1:

$$
\mathcal{W}_1(Y) = \{\omega \in \mathcal{W}(Y) : \mathbf{M}^{\dagger}(\omega) \leq 1\}.
$$

Definition 8.2. *For $S \in \mathbf{F}_m(Y)$, define the maximal flux through S as*

$$
\Phi^{\dagger}(S) = \sup\{\langle S, \omega \rangle : \omega \in \mathcal{W}_1(Y)\}.
$$

$\Phi^{\dagger}(S)$ will be finite if S is homologous to a member of $q_{\#}\mathbf{F}_m(W)$, otherwise it will generally be infinite. But this is good enough, since for our reference currents S we must assume a feasible exterior Q exists, whence $S + \partial Q \in q_{\#}\mathbf{F}_m(W)$.

Maximal flows always exist; no special assumptions are needed, except that W has finite pair diameter.

Theorem 8.3 (Existence of maximal flow). *Suppose W has finite pair diameter d and S is a reference surface. Then there is $\omega \in \mathcal{W}_1(Y)$ such that*

$$
\langle S, \omega \rangle = \Phi^{\dagger}(S) < \infty.
$$

Proof. Let $X \subset \mathbf{F}_m(Y)$ be the subspace of chains homologous to members of $q_\# \mathbf{F}_m(W)$. Then $\Phi^\dagger(R)$ is a positive convex homogeneous function of $R \in X$. Homogeneity is clear; convexity follows from:

$$\begin{aligned}
\Phi^\dagger(R + P) &= \sup\{\langle R, \lambda \rangle + \langle P, \lambda \rangle : \lambda \in \mathcal{W}_1(Y)\} \\
&\leq \sup\{\langle R, \lambda \rangle : \lambda \in \mathcal{W}_1(Y)\} + \sup\{\langle P, \lambda \rangle : \lambda \in \mathcal{W}_1(Y)\} \\
&\leq \Phi^\dagger(R) + \Phi^\dagger(P).
\end{aligned}$$

Note also that if $Z \in \mathbf{F}_{m+1}(Y)$ then $\partial Z \in X$ and $\Psi(Z) = 0$.

By the Hahn-Banach Theorem, there is a linear function $\omega : X \to \mathbb{R}$ such that

$$\begin{aligned}
\langle S, \omega \rangle &= \Phi^\dagger(S), \\
\langle \partial Z, \omega \rangle &= 0 \qquad \text{for } Z \in \mathbf{F}_{m+1}(Y) \\
\langle R, \omega \rangle &\leq \Phi^\dagger(R) \quad \text{for all } R \in X.
\end{aligned}$$

Since $\Phi^\dagger(R) \leq \mathbf{M}^\dagger(R) \leq d\mathbf{M}(R)$, where d is the pair diameter of W, one can use the Hahn-Banach Theorem again to extend ω to all of $\mathbf{F}_m(Y)$ with

$$\begin{aligned}
\langle S, \omega \rangle &= \Phi^\dagger(S), \\
\langle \partial Z, \omega \rangle &= 0 \qquad \text{for } Z \in \mathbf{F}_{m+1}(Y) \\
\langle R, \omega \rangle &\leq d\mathbf{M}(R) \quad \text{for all } R \in \mathbf{F}_m(Y).
\end{aligned}$$

These properties imply $\mathbf{M}(\omega) \leq d$ and $d\omega = 0$, so $\omega \in \mathbf{F}^k(Y)$ with $\mathbf{F}(\omega) \leq d$. Further, $\mathbf{M}^\dagger(\omega) \leq 1$ since $\langle Z, q^\# \omega \rangle \leq \Phi^\dagger(q_\# Z) \leq \mathbf{M}(Z)$ for $Z \in \mathbf{F}_k(W)$. Hence $\omega \in \mathcal{W}_1(Y)$. $\quad \square$

Corollary 8.4. *Under the assumptions of Theorem 6.1, the real minimization problem* (RMIN) *has calibrated solutions.*

Proof. By Theorem 6.1 solutions exist, and by Theorem 8.3 they are calibrated. $\square$

Remark. It is not true that any minimal film may be calibrated with a smooth flow. If that were true, then each type of singular tangent cone would have to be calibrated by a flow constant on each sheet. However, the cone over a hypercube in $\mathbb{R}^4$ cannot be so calibrated [B1]. The minimal octahedral film of example 8.8 is a probable example in $\mathbb{R}^3$.

§8.3. Calibration examples. It is very difficult in practice to calibrate arbitrary films, since exact formulas for films are hard to find. Some very simple examples are included here to show how calibration works in various circumstances.
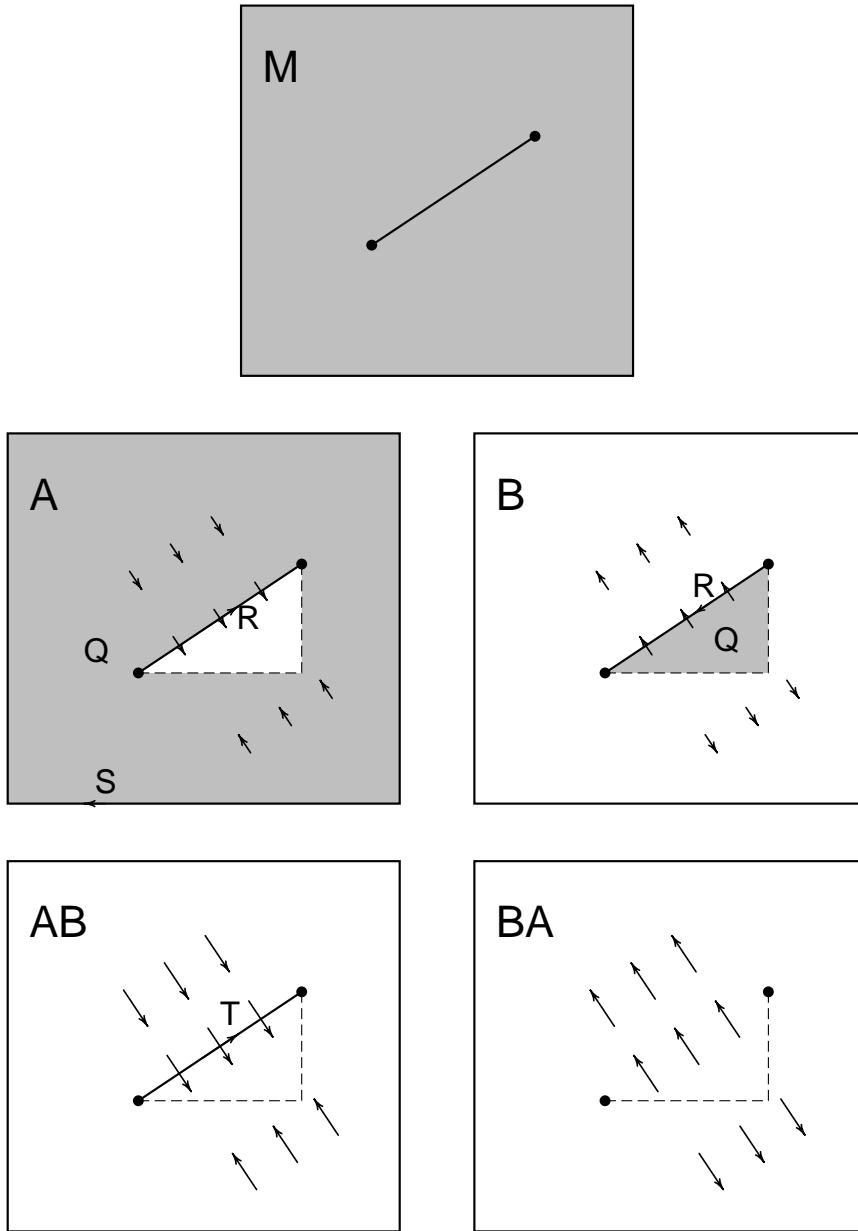
Figure 17. Calibrating a straight line between two points. The top row shows the base manifold M. The second row shows sheets A and B of the covering space Y with the reference current S. The dashed line is the cut for gluing the sheets together. The straight solid lines between the points are the soapfilm sides. Note the orientation arrows. The free-standing arrows show the flow. The bottom row shows the pair covering space W with the minimal mass representation and the difference flow. The flow arrows in Y have magnitude 0.5, so the differences have magnitude 1. (Example 8.5)

Example 8.5. Straight line between two points. As a first example to clarify concepts, consider the problem of showing that the one-dimensional soap film joining two points P_1, P_2 in the plane is a straight line. (No novelty is claimed for this result.) See figure 17 for an illustration. The ambient manifold is a large

square E and the boundary set consists of the two points P_1, P_2. So the complement manifold is $M = E - \{P_1, P_2\}$. Let the covering space Y consist of two sheets glued across any cut between P_1 and P_2. In figure 17, the cut has been deliberately chosen not to be the straight line between P_1 and P_2. The two sheets are labelled A and B for convenience. The pair covering space W happens to be isomorphic to Y, and has two sheets labelled AB and BA. The reference surface S is the outer boundary of sheet A, the lift of the boundary of M. We can easily exhibit the minimal chain and a maximal flow. The minimal chain ∂Q is an oriented line T_A from P_1 to P_2 in sheet A and a line T_B from P_2 to P_1 on B, both of density 1. One maximal flow on Y is a constant unit flow of magnitude 0.5 perpendicular to P_1 and flowing into both sides of the cut. There are many maximal flows, but they all have the property of having magnitude difference 1 on the minimal current. There are many possible choices for the film-mass representative T in W. Pictured is one that lives entirely on sheet AB. One could instead have take the negative of it on sheet BA, or any convex combination thereof.

Example 8.6. Tripod. Let P_1, P_2 and P_3 be three vertices of an equilateral triangle of circumradius 1 in the plane, as shown in figure 18. The goal here is to prove that soap film consisting of three segments joining the vertices to the center is minimizing for some covering space. Again, E is a large square and $M = E - \{P_1, P_2, P_3\}$. We will take Y to be a three-sheeted covering space of M such that each vertex is a branch point of order 3. In figure 18, the cuts are segments P_1P_2 and P_2P_3. The minimal chain ∂Q in Y consists of three 120° V-shapes, one on each sheet, and it is an integral chain. The minimal film-mass representative T consists of three unit-length segments of density 1 in W. Hence the film-mass of the tripod is 3. A maximal flow may be constructed by taking the flow on sheet A to be perpendicular to each side of the equilateral triangle with magnitude $1/\sqrt{3}$, extending to the various sheets. The difference magnitudes are 1 where the flows overlap, and the total flux is 3. Hence the tripod is absolutely minimizing. It is possible to construct a maximal flow that has difference magnitude 1 only on the tripod, showing that the tripod is the unique solution.
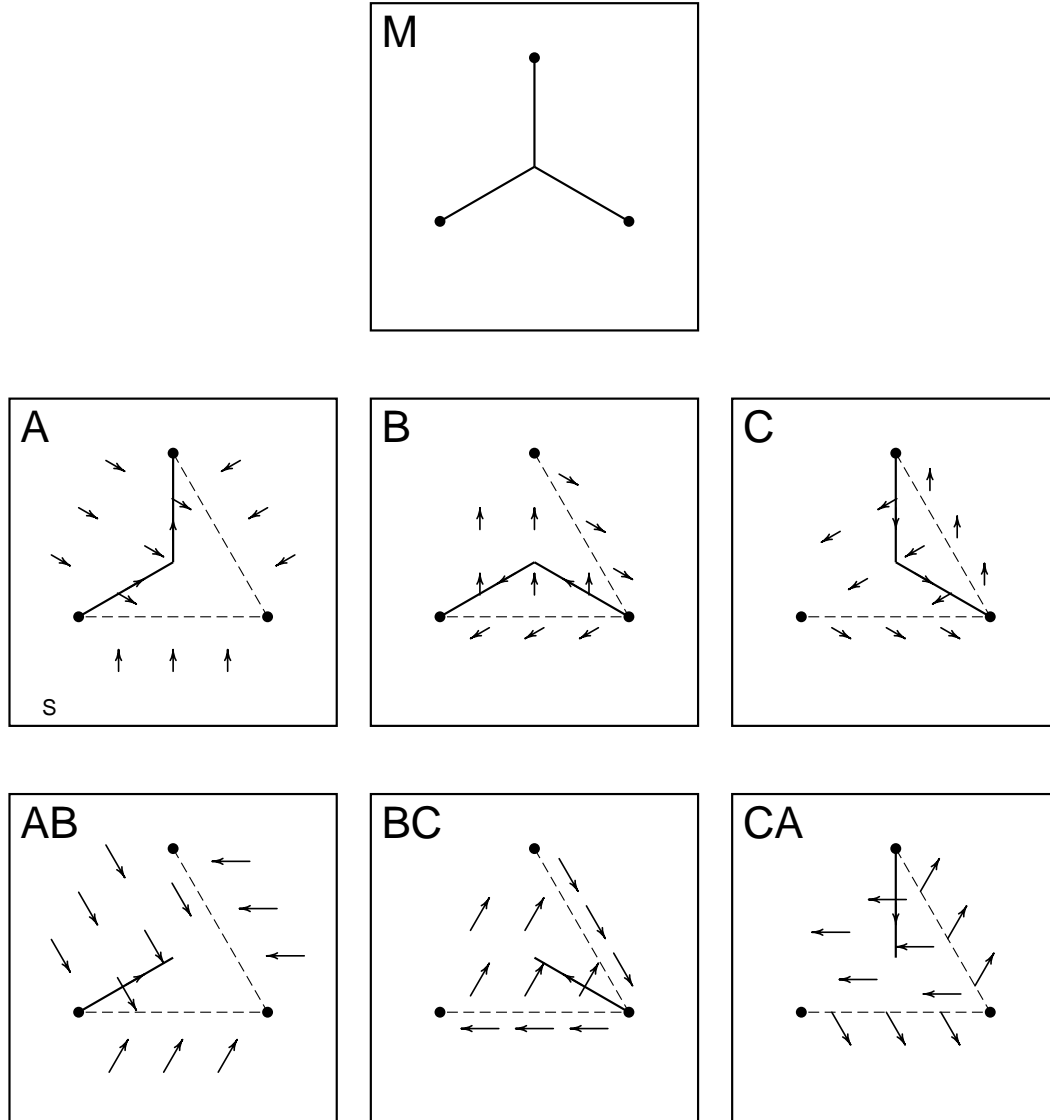
Figure 18. Tripod joining three vertices of an equilateral triangle. Top row shows the base space M.
The second row shows the sheets of Y and the reference surface S. Dashed lines are cuts, solid
lines the sides of the film. The bottom row shows the least mass representation and the difference flows.
(Example 8.6)

Example 8.7. Regular N-gon. Consider the boundary to be the vertices
of a regular N-gon in the plane (the previous example was $N = 3$). Let Y be
an N-sheeted covering in which sheet A is reciprocally glued to each of the other
sheets across a cut along a different polygon side. For $N = 4, 5$ the minimal film is
a network in the interior of the polygon, but for $N \geq 6$ the minimal film consists
of $N - 1$ of the polygon sides, as shown in the top of figure 19. In the latter case,
a maximal flow may be described as a flow of magnitude $(N - 1)/N$ crossing each
edge from sheet A, fanning out inside the polygon, and exiting across all the other
edges with magnitude $1/N$. Hence the total flux is $N - 1$ times the side length.
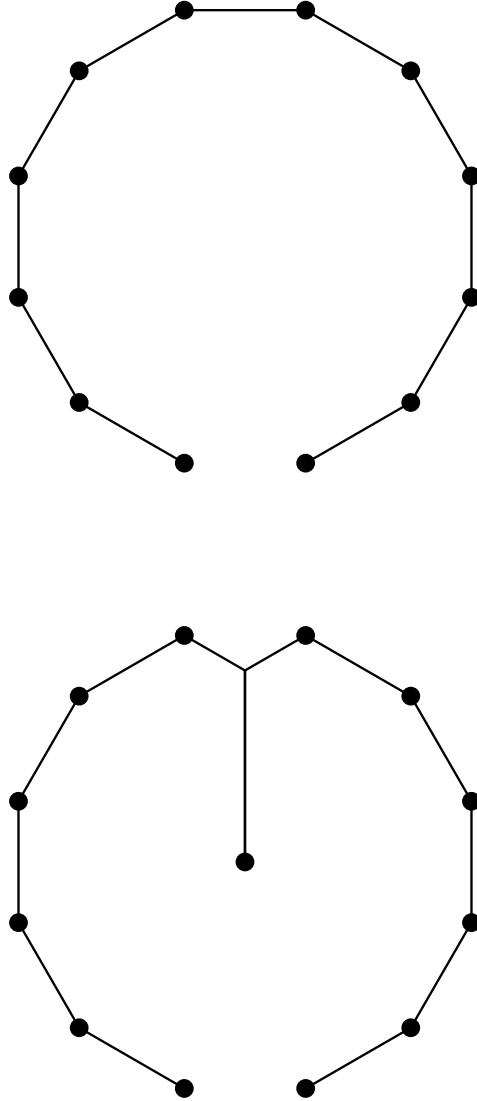
Figure 19. Soap film connecting the vertices of a polygon, without and with a central point. (Example 8.7)

A more interesting example can be made by putting another boundary point at the center of the polygon (still considering $N \geq 6$). The covering space has the added feature that one can reach additional sheets by spiralling around the center point. Suppose there is an m-fold increase in the number of sheets. Then each edge flux has one entrance and $mN - 1$ exits, so has magnitude $(mN - 1)/mN$. The total flux is $N - 1/m$ side lengths, and since the film must be confined to the polygon sides, the solution clearly cannot be an integral film. This is an example where integral initial data does not have an integral solution.

For $N \geq 12$, the film one would expect on this boundary should have $N - 1$ polygon sides and a radial segment from the center out to the middle of one side,

pulling that side in to form a 120° angle, as shown in the lower part of figure 19. The total length of the film is greater than the perimeter of the polygon, so the method of the last paragraph would not work. The way to handle this is to not take the full pair covering space, but to take W to be the component that included pairs on successive levels of Y but not more distant levels. That is, $(y_1, y_2) \in W$ iff a path from y_1 to y_2 projected to M has linking number ± 1 with the center boundary point. Then the flux could escape from the center after spiralling around the center a couple of times. The inward flux would be $1 + a$ on sheet A, a on the neighboring sheets of A, and outward on sheets further away. The new bottleneck would come from flux spiralling around the center, which gives the radial segment of film. A precise construction of such a covering space is as follows. Take N three-fold connected covers of the inside of the polygon. These are spirals. Label the polygon edges on each $1, \ldots, 3N$ in order. Also take $3N$ disjoint copies of the exterior of the polygon, labelled $1, \ldots, 3N$, with edges on each labelled $1, \ldots, N$. One of these will be sheet A with the reference surface. Glue edge j of outside sheet i to edge i of spiral j. The pair space W consists only of those pairs on the outside sheets separated by less than N sheets (mod $3N$) and the continuation of such to the spirals.

Example 8.8. Octahedral frame. It is a remarkable fact that an octahedral wire frame can bound at least five different soap films with simply connected complement. These are shown in figure 20. The one of these with the smallest area is the central one, made up entirely of plane films with a tetrahedral point in the center. Despite its seeming simplicity, I have not been able to come up with a calibration for it. The problem stems from the fact that it is possible to fit this film into the frame in two different positions, and a calibrating flow must calibrate both. Thus at the center, the flow must simultaneously calibrate two tetrahedral films in opposing orientations. It is possible that this is not the real flat chain minimum, that the real minimum is non-integral. Example 7.10 shows how this does seem to happen in two dimensions.
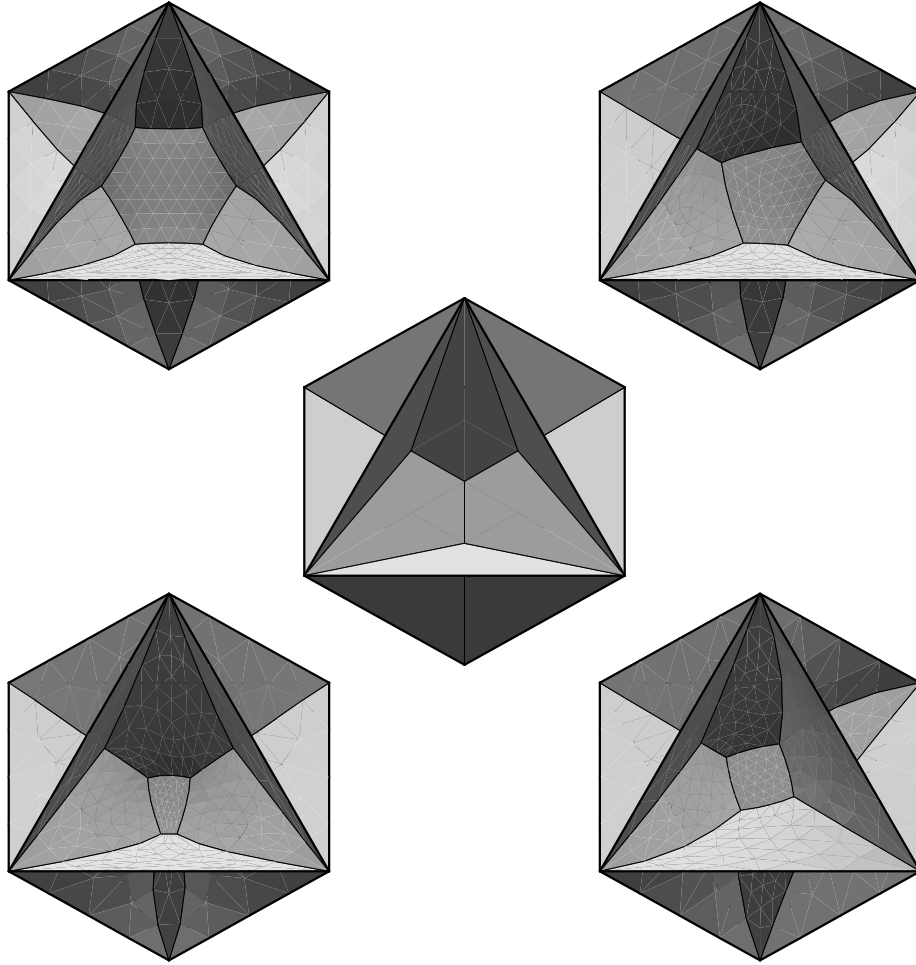
Figure 20. Five films that can form on an octahedral frame. (Example 8.8)

If no invisible wires are used, then eight sheets are needed. To see this, consider the cuts used in gluing the covering space together to be seven of the faces of the octahedron. Suppose less than eight sheets are used. Then from sheet A, two of the faces will open on the same sheet inside the octahedron. Hence no film will be needed to separate these two faces. But for any such pair, a minimal film can be put in with smaller area than the whole octahedral film.

9. Shadow prices and uniqueness

§**9.1. Shadow prices.** Compactness can show the existence of minimal films. However, for more control over their existence and uniqueness, one can look at calibration properties. The method is essentially the same as examining the solution set of a linear programming problem. The Hahn-Banach Theorem can give more control over the existence of solutions, and shadow prices can be used to prove uniqueness. The idea of shadow prices is that if the minimal film is unique, then loosening the flow bound at any point on the film will increase the maximum flux.

The marginal effect of loosening is defined in terms of the change in flux due to adjusting the flow bounds by an arbitrary flat chain:

Definition 9.1. *Suppose $S \in \mathbf{F}_m(Y)$ is a reference surface. For $\phi \in \mathbf{F}^m(W)$ and $\alpha \geq 0$ define*

$$J(\phi, \alpha) = \sup\{\langle S, \lambda \rangle : \lambda \in \mathcal{W}(Y), \mathbf{M}(q^{\#}\lambda - \alpha\phi) \leq 1\},$$
$$J'(\phi) = \lim_{\alpha \to 0^+} J(\phi, \alpha)/\alpha.$$

In linear programming terms, J' is the shadow price of the flow bound constraint. J and J' have some easily verified properties:

Lemma 9.2.

  (1) $J(\phi, 0) = \Phi^{\dagger}(S)$.
  (2) $J(\phi, \alpha)$ is concave in α, so $J'(\phi)$ is well-defined.
  (3) $J'(\phi)$ is concave in ϕ.
  (4) $|J'(\phi)| \leq \mathbf{M}(\phi)\Phi^{\dagger}(S)$.
  (5) If $\phi \in \mathcal{W}(Y)$, then $J'(q^{\#}\phi) = \langle S, \phi \rangle$.

$\square$

Next we establish that the shadow prices support the convex solution set.

Lemma 9.3. *Suppose $T \in \mathbf{F}_m(W)$ is a solution of the real minimization problem (RMIN) and $\phi \in \mathbf{F}^m(W)$. Then $J'(\phi) \leq \langle T, \phi \rangle$.*

Proof. If $\lambda \in \mathcal{W}(Y)$ and $\mathbf{M}(q^{\#}\lambda - \alpha\phi) \leq 1$, then

$$\langle S, \lambda \rangle = \langle T, q^{\#}\lambda \rangle = \langle T, q^{\#}\lambda - \alpha\phi \rangle + \langle T, \alpha\phi \rangle \leq \mathbf{M}(T) + \alpha\langle T, \phi \rangle.$$

Hence

$$J(\phi, \alpha) \leq \Phi^{\dagger}(S) + \alpha\langle T, \phi \rangle$$

and the conclusion follows. $\square$

The following theorem shows that minimal films do indeed fill out the convex set defined by the shadow prices.

Theorem 9.4 (Calibrated minimal film). *Suppose Y is flat-m-bounded and flat-k-compact, and S is a reference surface. If $\phi \in \mathbf{F}^m(W)$ then there exists $T \in \mathbf{F}_m(W)$ such that T is a solution of the real minimization problem (RMIN) for S and $\langle T, \phi \rangle = J'(\phi)$.*

Proof. By the Hahn-Banach Theorem and Lemma 9.2(3)(5), there is a linear function $L : \mathbf{F}^m(W) \to \mathbb{R}$ such that

  (1) $\langle L, \phi \rangle = J'(\phi)$,
  (2) $\langle L, q^{\#}\lambda \rangle = \langle S, \lambda \rangle$ for all $\lambda \in \mathcal{W}(Y)$,
  (3) $\langle L, \lambda \rangle \geq J'(\lambda)$ for all $\lambda \in \mathbf{F}^m(W)$.

This L is a flat cochain. By (3) and Lemma 9.2(4), $\mathbf{M}(L) \leq \Phi^{\dagger}(S)$. Let T be the restriction of L to $\mathcal{D}^m(W)$. Since $\mathbf{M}(T) \leq \mathbf{M}(L)$, $T \in \mathcal{D}_m(W)$. By (2) and Lemma 9.2(5), $q_{\#}T \sim S$, so $\partial q_{\#}T = 0$ and $q_{\#}T$ is normal. For $\phi \in \mathcal{D}^m(Y)$,

$$\langle q_{\#}T, \phi \rangle = \langle T, q^{\#}\phi \rangle = \langle L, q^{\#}\phi \rangle.$$

By Proposition 3.10, this extends to all flat cochains, so

$$\langle q_{\#}T, \lambda \rangle = \langle S, \lambda \rangle \qquad \text{for all } \lambda \in \mathcal{W}(Y).$$

Flat-k-boundedness implies $q_{\#}T \approx S$. Therefore, T is a feasible solution, and $\mathbf{M}(T) = \Phi^{\dagger}(S)$. $\square$

Remark. If Y is infinite-sheeted (hence not flat-m-compact), then the film T shown to exist in the above proof may not be double-sided. Consider the N-gon with central point of example 8.7 with $m = \infty$. The cocochain L essentially is a polygon with density 1 on sheet A and density $1/\infty$ on infinitely many other sheets. Hence the current restriction is just the single-sided film on sheet A. It is a double-sided cocochain, but not a double-sided chain.

§**9.2. Uniqueness.**   Having a characterization of the solution set in terms of shadow prices, we can formulate some criteria for uniqueness of solutions.

Theorem 9.5 (Uniqueness in W). *Suppose $T \in \mathbf{F}_m(W)$ is an optimal solution to the real minimization problem* (RMIN). *If for every $\phi \in \mathcal{D}^m(W)$ we have*

$$\langle T, \phi \rangle = J'(\phi),$$

then T is unique. Conversely, if Y is flat-k-bounded and T is unique, then for every $\phi \in \mathbf{F}^m(W)$ we have

$$\langle T, \phi \rangle = J'(\phi).$$

Proof. Suppose that T is not unique, that there is another minimal T_1. Then there is some $\phi \in \mathcal{D}^m(Y)$ with $\langle T, \phi \rangle = 0$ but $\langle T_1, \phi \rangle < 0$. Then by Lemma 9.2(4), $J'(\phi) < 0$ and so $J'(\phi) \neq \langle T, \phi \rangle$.

Conversely, if there is $\phi \in \mathbf{F}^m(Y)$ with $J'(\phi) \neq \langle T, \phi \rangle$, then by Theorem 9.4 T is not unique. $\square$

The same style argument applies in Y:

Theorem 9.6 (Uniqueness in Y). *Suppose $R \in \mathbf{F}_m(Y)$ is an optimal solution to the real minimization problem. If for every $\phi \in \mathcal{D}^m(Y)$ we have*

$$\langle R, \phi \rangle = J'(q^{\#}\phi),$$

then R is unique. Conversely, if Y is flat-k-bounded and R is unique, then for every $\phi \in \mathbf{F}^m(Y)$ we have

$$\langle R, \phi \rangle = J'(q^{\#}\phi).$$

   $\square$

§**9.3. Uniqueness test.**   In practice, one would like a test that is easier to apply than calculating all possible shadow prices. If the film is reasonably well-behaved, then one may be able to construct a calibration that calibrates only one possible film.

Theorem 9.7(Uniqueness test). *Suppose* (Q, R, T) *is a solution for the real problem* (RMIN). *Suppose* $\omega \in \mathbf{F}^m(Y)$ *is a maximal flow. Define the critical points in* W *as*

$$C_W = \{w \in W : ||q^{\#}\omega(w)|| = 1\}.$$

Suppose R *is a rectifiable current and*

$$spt(R) = |R| = q_1(C_W) \cup q_2(C_W).$$

where $|R|$ *is the oriented rectifiable set associated to* R. *Suppose that each stream-line of* ω *passes though only one point of* $|R|$. *Then the minimal film* R *is unique.*

Proof. Suppose $(\hat{Q}, \hat{R}, \hat{T})$ is another minimal film. Then ω must calibrate it. Hence $\hat{R}$ must have support in $|R|$, and hence by [F1,4.1.20] $||\hat{R}||$ is continuous with repect to $\mathcal{H}^m \mathbin{\llcorner} |R|$. Thus $\hat{R}$ be of the form $R \mathbin{\llcorner} f$ for some $||R||$-measurable density function $f(y) : |R| \to \mathbb{R}$. That the density function is uniquely determined may be seen by considering test flows ϕ constructed using the streamlines of ω. In particular, if $g(y) : |R| \to \mathbb{R}$ is a scalar test function on $|R|$, then one can construct a test flow ϕ by multiplying the flow on each streamline of ω by the value of g where the streamline hits $|R|$. Then

$$\langle \hat{R}, \phi \rangle = \int_{|R|} g(y) f(y) \langle \omega(y), \vec{R} \rangle \, d\mathcal{H}^m y = \langle S, \phi \rangle,$$

so the density is uniquely determined.   □

Remark. The minimal mass representative T in W need not be unique even if R is, and it will not be if W contains the opposite pair BA for every pair AB.

Example: Consider the straight line in example 8.5. By constructing a flow that fans out and has magnitude difference less than 1 off the minimal chain, one can show the minimal chain is unique in Y, although as noted in example 8.5 the film-mass representative T in W is not unique.

§9.4. Solution set. If the solution is not unique, then the set of solutions forms a convex set. The most convenient way to specify this set is often to give the set $\mathcal{T}$ of extreme points, so the solution set is the closed convex hull of $\mathcal{T}$. The following proposition gives a criterion for having a suitable $\mathcal{T}$:

Theorem 9.8 (Solution set). *Suppose* $\mathcal{T} = \{T_\sigma : \sigma \in \Sigma\}$ *is a set of optimal solutions. Then the flat-closed convex hull of* $\mathcal{T}$ *is the set of all optimal solutions iff*

$$J'(\phi) = \inf\{\langle T_\sigma, \phi \rangle : \sigma \in \Sigma\} \text{ for all } \phi \in \mathbf{F}^m(Y).$$

Proof. Suppose there is an optimal solution T not in the flat-closed convex hull of $\mathcal{T}$. Then by the hyperplane separation property of convex sets, there is $\phi \in \mathbf{F}^m(Y)$ such that

$$\langle T, \phi \rangle < \inf\{\langle T_\sigma, \phi \rangle : \sigma \in \Sigma\}.$$

Then by Lemma 9.2,

$$J'(\phi) \le \langle T, \phi \rangle < \inf\{\langle T_\sigma, \phi \rangle : \sigma \in \Sigma\}.$$

Conversely, suppose there is a ϕ with

$$J'(\phi) < \inf\{\langle T_\sigma, \phi \rangle : \sigma \in \Sigma\}.$$

By theorem 9.4, there is an optimal solution T with $\langle T, \phi \rangle = J'(\phi)$, so T is not in the flat-closed convex hull of $\mathcal{T}$. $\square$

10. REGULARITY

§10.1. $(\mathbf{M}, 0, \delta)$-minimality.   The general regularity theory of covering space films remains to be worked out. However, in a common case of integral films, the support of a film projected to M is an $(\mathbf{M}, 0, \delta)$-minimal set and so inherits all the regularity thereof.

Lemma 10.1. *Suppose the pair covering space is the full pair covering space. If $R \in \mathbf{I}_m(Y)$ is a feasible solution to the integral problem (IMIN), then $\mathbf{M}^\dagger(R) = \mathcal{H}^m(p(|R|))$, where $|R|$ is the rectifiable set associated with R.*

Proof. Let $Q \in \mathbf{F}_{m+1}(Y)$ be the integral current for the exterior of R. R is an integral current of finite mass and no boundary, and hence may be represented as the current corresponding to a rectifiable set $|R|$. Hence the lemma reduces to considering the film-mass pointwise on $p(|R|)$. At $\mathcal{H}^m$ almost all points x of $p(|R|)$, the tangent cone structure of Q above x will consist of two half-disks on different sheets. This pair is in the pair covering space by assumption, so the film-mass is equal to the Hausdorff measure of the projection of $|R|$.

Theorem 10.2. *Suppose W is the full pair covering space, and suppose R is a solution to the integral minimization problem. Then $p(|R|)$ is $(\mathbf{M}, 0, \delta)$-minimal.*

Proof. Let $Q \in \mathbf{F}_{m+1}(Y)$ be the integral current for the exterior of R. Suppose there is a deformation f in small disk of M that decreases the area of $p(|R|)$. Then this deformation lifts to Y and W. The deformed $f_\# Q$ consists of a countable number of pieces of constant integral density whose boundaries lie over $f(p(|R|))$. One may choose a selection of these pieces with density $+1$ to form a Q' which covers M. Let $R' = \partial Q' + S$. By the previous lemma, $\mathbf{M}^\dagger(R') \leq \mathcal{H}^m(f(p(|R|)))$. But Q' is a feasible solution, so $\mathcal{H}^m(f(p(|R|))) \geq \mathcal{H}^m(p(|R|))$. Thus $p(|R|)$ is $(\mathbf{M}, 0, \delta)$-minimal. $\square$

If W is not the full pair covering space, then $p(|R|)$ need not be $(\mathbf{M}, 0, \delta)$ minimal. A counterexample in $\mathbb{R}^2$ can be constructed by taking the superposition of two line segments to form a cross, following the procedure described in section 12.5. The primary covering space has four sheets A,B,C,D, and the pair covering space has the four sheets AB, BC, CD, and DA. If W had all pairs, then the cross would split into two triple points, and the segment between the triple points would have film density 1, as a film on sheet AC of W. But since AC is not included, the film density of the segment would be 2, requiring film on sheets AB and BC, say. This higher tension is enough to pull the triple points together into a quadruple point.

The necessity of the integral nature of Q is shown by example 7.10. Here all pairs are included in the pair covering space, but the quadruple and pentuple points shown in the top right diagram of figure 16 are legal singularities and their tangent

cones can be calibrated with constant vector fields. The previous theorem does not apply because Q is permitted to exist on two sheets simultaneously.

Regularity almost everywhere in the case of a partial pair covering space is beset by the "bubbling problem." The film-mass is a multiple of the Hausdorff measure of the support. Consider a point on the projected support with a tangent plane where the film-mass density is greater than 1. Call the two sheets on opposite sides of the tangent plane where Q is present A and B. The film may be thought of being made of lamina in number equal to the pair distance between A and B. To prove regularity almost everywhere, one has to prove that these lamina don't delaminate on a dense set of small measure, i.e. the bubbles between lamina are not dense in a set of larger measure. The difficulty is that having a tangent plane at a point is not enough to eliminate nearby bubbles. A counterexample can be formed as follows. The film will consist of two order 4 saddles (approximately $z = r^4 \sin 4\theta$), each a minimal surface, superposed so they are tangent at the center but rotated so one surface's hills are above the other's valleys. Section 12.5 shows how to construct the superposition of two minimial films so the superposition is minimal. Then there is a tangent plane to the film at the origin, but the film is not regular there.

One can say that there is a minimal varifold associated to the film.

Theorem 10.3. *If (Q, R, T) is a solution to the real or integral film problems, and V is the varifold associated with R, then $p_\# R$ is a minimal varifold in M.*

Proof. Suppose f is a deformation of M that reduces the mass of $p_\# V$. Then f lifts to deformations in Y and W with $f_\# p_\# V = p_\# f_\# V$ since there is no cancellation with varifolds. Since $f_\# T$ is a possible representative of $f_\# R$, we must have

$$\mathbf{M}^\dagger(f_\# R) \leq \mathbf{M}(f_\# T) \leq \mathbf{M}(f_\# V) = \mathbf{M}(p_\# f_\# V) \leq \mathbf{M}^\dagger(R).$$

Hence by the minimality of R, there can be no such mass reducing deformation, and $p_\# V$ is a minimal varifold.

§**10.2. Integrality of real solutions.** The calibration arguments apply to real currents. If the integral minimizer is also a real minimizer, then calibration arguments will work for the integral minimizer. If there are multiple integral minimizers (e.g. spanning the corners of a square) then any convex combination of integral solutions is a real solution. The interesting question is whether there are any real minimizers that are not convex combinations of integral solutions. Examples 7.10 (crossed tripods) and 8.7 (polygon) are cases of this. Examples 8.8 (octahedral frame) is a case where where this probably happens. If the integral solution is not the real solution, then proving the minimal integral solution becomes much tougher, essentially the difficulty in going from linear programming to integer programming.

It is known in classical real flat chain minimization that in codimension 1 with integral reference surfaces that there are integral minimizers [F2,5.10]. Perhaps it can be proven that if the reference surface S is integral, then the extreme points of the solution set are rational multiples of integral currents, with denominators depending on the number of sheets. It can be shown that if Y is a double cover and S is integral, then there is a solution that is at worst half-integral.

Theorem 10.4. *Suppose Y is a double cover of M, and the reference surface $S \in q_{\#}\mathbf{I}_m(Y)$ is the boundary of one sheet. Then there is a solution $R \in \mathbf{F}_m(Y)$ of the real minimization problem such that $2R \in \mathbf{I}_m(Y)$.*

Proof. The idea is to show that classical solutions give rise to film solutions. In the case of a double cover, film-mass is exactly half the mass for double-sided chains $R \in q_{\#}F(W)$. The classical real minimization problem is

$$
\begin{aligned}
&\text{For} &&R \in \mathbf{F}_m(Y) \\
&\text{minimize} &&\Psi = \mathbf{M}(R)/2 &&\text{(CRMIN)} \\
&\text{such that} &&R \approx S.
\end{aligned}
$$

Clearly any solution R of (RMIN) is also a feasible solution of (CRMIN). Conversely, suppose R is an optimal solution of (CRMIN). By [F2,5.10], we may take $R \in \mathbf{I}_m(Y)$. Let Q be the exterior chain for R, so $R = \partial Q + S$. Let $\sigma : Y \to Y$ be the involution that exchanges sheets of Y, preserving orientation. Define the double-sided chain

$$
\hat{Q} = (Q + \sigma_{\#}([\![Y]\!] - Q))/2.
$$

One checks that

$$
p_{\#}\hat{Q} = (p_{\#}\sigma_{\#}[\![Y]\!])/2 = [\![M]\!].
$$

By hypothesis on S, $\partial[\![Y]\!] = -(S + \sigma_{\#}S)$. Hence

$$
\begin{aligned}
\hat{R} = \partial\hat{Q} + S &= (\partial Q + \sigma_{\#}\partial[\![Y]\!] - \sigma_{\#}\partial Q)/2 + S \\
&= (R - S - \sigma_{\#}S - S - \sigma_{\#}R + \sigma_{\#}S)/2 + S \\
&= (R - \sigma_{\#}R)/2.
\end{aligned}
$$

Hence $\hat{R}$ is a feasible solution to (RMIN) with $\mathbf{M}(\hat{R}) \leq M(R)$. Hence $\hat{R}$ must be an optimal solution of (RMIN), and $2\hat{R} \in \mathbf{I}_m(Y)$. $\square$

A physical interpretation of nonintegral solutions with integral S might go as follows. View the film as separating regions with different fluids. Suppose the pure fluids can form mixtures whose surface tensions with each other are linearly proportional to the differences in composition. Then non-integrality would appear as regions of mixtures between the pure fluids.

11. Construction of films

It is very rare to be able to explicitly present a minimal film and a calibration for it. However, there are ways to control a film and get information about it. This section describes several such techniques.

§11.1. **Bounds on area.** Any feasible film provides an upper bound for area. In practice, one might generate such films with a computer program such as the Surface Evolver [B3], which represents films as simplicial complexes.

Any feasible flow provides a lower bound. Maximizing flow is a convex programming problem that can be discretized and solved by computer. I have begun work on a program to do this. The covering space Y is represented as a simplicial

complex. The independent variables are the scalar fluxes across simplex faces (and the flow is taken to be constant in simplices). The objective function is the total flux across the simplex faces making up S. The constraints are 1) divergenceless flow: total flux into a simplex is zero, and 2) unit bound on the magnitude of the difference in flow on pairs of sheets. When the solution is reached, the soap film is located at spots where the bound constraint has non-zero shadow price with direction orthogonal to the bound direction, and the total mass of the film is the total of the shadow prices. My program works for simple plane cases (like the tripod), but it is slow. The efficient standard network min-flow algorithm doesn't work because of the nonlocal bounds on the differences in flow magnitudes.

§**11.2. Location.** Any path from the closed boundary of sheet A to the closed boundary of another sheet must intersect the film, although one may not be able to say exactly where. One can hope to establish the approximate location of the film by finding "barriers", surfaces that confine the soap film. An obvious barrier is the convex hull of the boundary set. More precisely, one can take the convex hull of the branch curves of Y as seen from sheet A. This approach can be used in the partial knot film of example 7.4 to show that the film does not touch the entire knot. One can also use minimal surfaces as barriers, even if these don't come close to the actual surface.

Theorem 11.1(Barriers). *Suppose H_M is a region of M such that there is an m-cochain ν such that ν calibrates ∂H_M and $||\nu(x)|| < 1$ for $x \in M - H_M$. (This implies ∂H_M has nonnegative mean curvature in a generalized sense.) Let $H_Y = p^{-1}(H_M)$, and suppose the reference surface S is homologous to a current $\hat{S} \in \mathbf{F}_m(Y)$ with $spt(\hat{S}) \in \partial H_Y$. Then any minimizing real or integral film R has $spt(R) \subset H_Y$.*

Proof. Under the hypotheses, one can construct a film-mass decreasing projection of the exterior of H_Y to ∂H_Y that preserves homology and integrality. The projection is done by mapping along the streamlines of ν. To be precise, let $\mu = p^{\#}\nu$. For any $Z \in \mathbf{F}_m(U)$ define the projection $\pi_{\#}Z$ as for any scalar function $f(y) \in \mathbf{F}_0(Y)$ with $df \wedge \mu = 0$

$$\langle \pi_{\#}Z, f \wedge \mu \rangle = \langle R, f \wedge \mu \rangle.$$

There is a corresponding projection in W. It is easily seen that if Z is integral, then $\pi_{\#}Z$ is also integral. The projection reduces film-mass: Let $\phi \in \mathbf{F}^m(W)$ with $\mathbf{M}(\phi) \leq 1$. Then outside H_W, ϕ may be altered to a form $f \wedge \mu$ by matching flux on ∂H_W with $f \leq 1$. Then

$$\langle \pi_{\#}Z, \phi \rangle = \langle Z, \phi \rangle.$$

Hence

$$\mathbf{M}(\pi_{\#}Z) \leq \mathbf{M}(Z) - \int_{W - H_W} (1 - ||\mu||) \, d||Z||.$$

It remains to establish that $\pi_{\#}Z$ is homologous to Z. Suppose $\lambda \in \mathbf{F}_m(Y)$ and $d\lambda = 0$. Define the barrier flow multiplier f so that $f(y)$ is the flux of λ for $y \in \partial H_Y$. Let

$$\sigma(y) = \begin{cases} \lambda(y) & \text{for } y \in H_Y, \\ f(y) \wedge \mu(y) & \text{for } y \notin H_Y. \end{cases}$$

Then

$$\langle R, \lambda \rangle = \langle \hat{S}, \lambda \rangle \text{ since both } R \text{ and } \hat{S} \text{ are homologous to } S$$
$$= \langle \hat{S}, \sigma \rangle \text{ by construction of } \sigma$$
$$= \langle R, \sigma \rangle \text{ since both } R \text{ and } \hat{S} \text{ are homologous to } S$$
$$= \langle \pi_\# R, \sigma \rangle \text{ by definition of the projection}$$
$$= \langle \pi_\# R, \lambda \rangle \text{ by construction of } \sigma$$

$\square$

Corollary 11.2. *If H is the convex hull of the visible branch curves in sheet A, then the minimal film is inside H.* $\square$

This implies that the film cannot "hide" on sheets behind A. In particular, the film must be inside the convex hull of the boundaries visible from sheet A.

Many of the examples show films around holes through which the background sheet seen is sheet A. One way to demonstrate these holes really exist is to use a catenoid as a barrier. Suppose region H consists of the intersection of the slab between two parallel planes and the exterior of a catenoid whose axis is perpendicular to the planes. Further, suppose that the area of the catenoid part of the boundary of H has less area than the disks cut out from the planes. Then a flow as required by the Barrier Theorem can be constructed. Hence if H can be situated so that the exterior of H is entirely in sheet A, then any minimal current must be inside H.

It is not always true that if the background sheet is A, then there is no film in the line of sight. A simple counterexample is the catenoid viewed from the side.

Triple junctions make it tricky to get good barriers for general films. The big challenge is to come up with the proper barrier theorem that can really confine soap films to where they are supposed to be. The method of attack would be to show that any maximal flow could be modified to have difference magnitude less than 1 outside the barrier, so the film couldn't be there.

§11.3. Knotted boundaries. It is now easy to generalize the results of [PH] and [DW] to higher dimension:

Theorem 11.3. *Let B be a compact $(m-1)$-manifold in $\mathbb{R}^{m+1}$ with finite area that is knotted in the sense that its complement has a nontrivial fundamental group. Then B supports a minimal film that does not touch all of B.*

Proof. Let $M = \mathbb{R}^{m+1} - B$ be the complement of the knot. Pick a halfspace H such that $M - H$ has fundamental group $\pi(M - H)$ that is just $\mathbb{Z}$. Let G be a subgroup of $\pi(M)$ containing $\pi(M - H)$ such that $\pi(M)/G$ is finite. Let Y be the covering space of M corresponding to G. Then the number of sheets of Y is the order of $\pi(M)/G$. Form the reference current S by first making a cone S_0 in M from a point to B. Then $M - S_0$ is simply connected, and hence may be lifted to Y to form sheet A. Let S be the boundary of this lift. Since B has finite area, S has finite mass, so the minimization problem is feasible. The minimal area is nonzero, since Y has a nontrivial fundamental group. So by the Existence Theorem, there is a nontrivial minimal integral film T homologous to S. The boundary of H forms a barrier, so T does not touch B outside of H. $\square$

Example 11.4. A boundary need not be knotted to be partially touched by a soap film. Figure 21 shows an example due to Almgren [AF1, fig. 1.9] of an unknotted loop that supports a soap film only partially touching it. Clearly the proof of the previous corollary does not apply here. In fact, such films are retracts to the boundary.
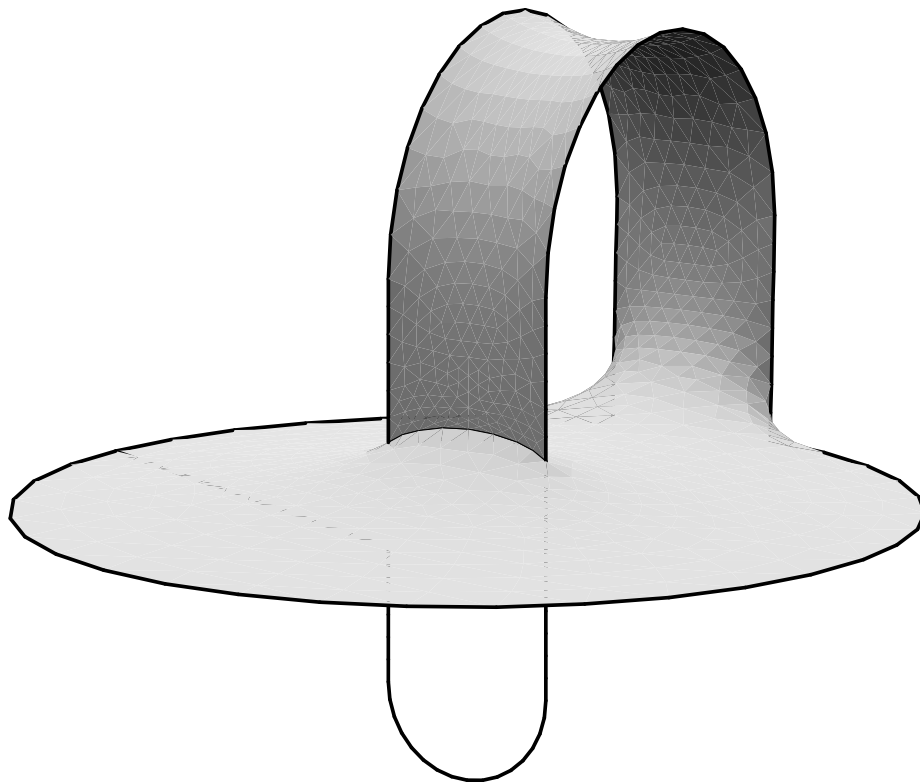

Figure 21. A soap film that partially touches an unknotted boundary. (Example 11.4)

Proposition 11.5. *If $B \subset \mathbb{R}^{m+1}$ is an unknotted image of S^{m-1} and $T \subset \mathbb{R}^{m+1}$ is a compact set with $B - T \neq \emptyset$, then there is a retract of T to B.*

Proof. Suppose $x \in B - T$. Since B is unknotted, there is an isotopy of $\mathbb{R}^{m+1}$ transforming B into a sphere and mapping x to the origin. Spherical inversion $x \to x/|x|$ maps the sphere to a $(m-1)$ plane, and T maps to a compact set. The desired retract is now just orthogonal projection to the plane. $\square$

The implication of this proposition is that to construct films partially supported by unknotted loops, invisible wires will always be necessary. In figure 21, one can use an invisible wire passing through the upper loop formed by the film.

12. Designing covering spaces

§12.1. Covering space for given soap film. Suppose one has a soap film and one wants to construct the appropriate covering space. The following algorithm seems to work well, although it constructs a covering space only of a neighborhood

of the film and it is not proven to work in all cases. Start with a closed boundary set B and suppose the film is initially represented as a set T. Soap films are stable, so pick a smooth closed neighborhood E of the film in which it is supposedly absolutely minimizing. Note this is not necessarily a neighborhood of B; if the film is not supported by all of B, then B is likely to extend outside of E. The base manifold is $M = E - B$. Take the reference surface $S \subset A$ to be the boundary of E. Construct a covering space by a "coloring algorithm". The idea is to treat the film as the cut between sheet A and the other sheets. The the two sides of the film will be "colored" according to which sheet is reached by going from sheet A through that side of the film. The constraint on coloring is that the three sides at a triple point or the four sides at a tetrahedral point must get distinct colors. The algorithm starts by assigning colors arbitrarily to the different sides at some point (a triple point or tetrahedral point if there is one). Then the color is propagated along the side of the film until it meets another color, it reaches the opposite side of the film from itself, or everything possible is colored. If it meets another color, say red meets green, then the sheets corresponding to those colors must be cut there and cross-glued. If this cut cannot be extended to the boundary of E without cutting other parts of the film, then an invisible wire may be introduced to terminate the cut. The reference surface is just the boundary of E.

This algorithm constructs a covering space with 2, 3, or 4 additional sheets besides sheet A, depending on what types of singular points exist. Actually, any two of these extra sheets may be identified and the covering will still work. This suggests the conjecture that any film may be calibrated in a covering space with at most four sheets. Even the full film on the spiral of example 7.9 can be calibrated in a four-sheeted covering space, although it was originally in an infinite-sheeted cover. However, if invisible wires are not used, then more than four sheets can be proved necessary. See the octahedron example 8.7.

One question that the conjecture does not cover is whether any covering space calibration may be reduced to some bounded number of sheets with the same solution set, since the domain is restricted there to a small neighborhood of one particular film. Consider the polygon with central point of example 8.6. Can all solutions be accommodated with a three-sheeted cover? Is there a bound on the number of sheets needed independent of the number of vertices?

The number of sheets needed must increase with dimension. Since the cone over hypercube frame is absolutely minimizing in all dimensions $N \geq 4$, the number of sheets must be at least $2N$, since the hypercube cone has $2N$ faces.

§12.2. **Oriented manifold spanning smooth boundary.** By theorems of Federer [F1] and Hardt and Simon [HS], if B is an oriented smooth embedded $m-1$ manifold in $\mathbb{R}^{m+1}$, then B is the boundary of an area minimizing integral current H such that $spt(H)$ is a smooth manifold with boundary at every point of B and elsewhere is a smooth manifold with singularities of Hausdorff dimension at most $m-7$. In particular, every smooth closed curve in $\mathbb{R}^3$ bounds an embedded oriented minimal manifold. Such a surface for the trefoil knot was shown in example 7.4. The existence and calibration of such surfaces can be done without covering spaces.

Recalling that E is M with B, the classical formulation is

$$
\begin{aligned}
\text{For} \quad & H \in \mathbf{F}_m(E) \\
\text{minimize} \quad & \Psi = \mathbf{M}(H) \\
\text{such that} \quad & \partial H = B
\end{aligned}
$$

with the dual calibration problem

$$
\begin{aligned}
\text{For} \quad & \omega \in \mathbf{F}^{m-1}(E) \\
\text{maximize} \quad & \Phi = \langle B, \omega \rangle \\
\text{such that} \quad & \mathbf{M}(d\omega) \le 1.
\end{aligned}
$$

But for the sake of completeness, let us see how they can be done with the covering space model.

Construct the covering space Y from an infinite set of copies of $\mathbb{R}^{m+1}$, indexed by the integers. Index 0 will correspond to sheet A. The cut surface G will be a cone over B with vertex in some nice position. Then G is an oriented surface with sides we shall designate positive and negative. Glue the sheets together so that entering the positive side of G goes to the next higher indexed sheet and entering the negative side goes to the next lower indexed sheet. The pair covering space W contains only pairs (j,j+1) of adjacent sheets. The singularity spaces Z_i are empty.

$$
\begin{aligned}
\text{For} \quad & Q \in \mathbf{F}_{m+1}(Y), T \in \mathbf{F}_m(W) \\
\text{minimize} \quad & \Psi = \mathbf{M}(T) \\
\text{such that} \quad & q_\# T - \partial Q = S \\
& \partial T = 0.
\end{aligned}
$$

The classical minimizing integral current H clearly gives rise to a feasible film when its exterior is lifted to Y, since any closed path linking B must intersect H. Conversely, any feasible integral film T in W can be mapped to an integral current in $\mathbb{R}^{m+1}$ by direct projection $p_\# q_{1\#}$ from W to $\mathbb{R}^{m+1}$. Since there are no homological singularities, $p_\# q_{1\#} T$ has no boundary other than B. Further, all of B is on the boundary of $p_\# T$ since any part of B provides an exit path from sheet A.

§**12.3. Obstacle problems.** In an obstacle problem, the additional constraint is made that the film be outside some obstacle set $\Omega \subset M$. For an example of a "thin obstacle" problem, where the obstacle has dimension $m-1$, see example 7.5. Here we will assume a "thick" obstacle Ω, which is an open subset of M. The problem is set up as usual, but with the additional constraint that $p(spt(R)) \cap \Omega = 0$. This is a well-behaved constraint in the existence theorems. The effect is the same as using an infinite mass for the film in the obstacle. In the dual maximizing flow problem, one gets that $q^\# \omega$ is unbounded in the obstacle.

§**12.4. Suppressing singularities.** It may be desired to produce a film that has no singularities, or singularities of only certain types. Here we discuss two ways to suppress undesired homological singularities. (There still may be geometric singularities, though).

The first method is by a judicious choice of Y and W so that the higher singularity spaces $Z_3, Z_4, \ldots$ do not have the undesired types of singularities. Section 12.2 uses this approach.

The second method, which eliminates all homological singularities, is to add the constraint $\partial T = 0$ to the problem statement. This constraint is well-behaved with regard to the existence proofs. The effect on the dual maximal flow problem is to replace the constraint

$$||q^{\#}\omega|| \leq 1$$

with the constraint that there is a cochain $\mu \in \mathbf{F}^{m-1}(W)$ such that

$$||q^{\#}\omega - d\mu|| \leq 1.$$

In working out examples, it is important to remember that μ has no flux on the boundary B.

§12.5. Superposing films. Given two different minimal films on the same base manifold, with possibly different covering spaces, there is a construction that gives a product covering space in which the film is essentially the superposition of the two films. Let one film have covering spaces Y_1 and W_1 with Roman indices to label sheets, and the other Y_2 and W_2 with Greek indices. For the product Y, take a set of sheets $i\alpha$ labelled by the product of the label sets. The fiber over each point of the base manifold is the product of the fibers of Y_1 and Y_2. The home sheet is labelled 00. The reference surface S is the outer boundary of the home sheet. For the pair covering space, take all pairs of the form $(i\alpha, j\alpha)$ where (i, j) is a pair in W_1 and all pairs of the form $(i\alpha, i\beta)$ where (α, β) is a pair in W_2. Suppose $T_1 \in \mathbf{F}_m(W_1)$ and $T_2 \in \mathbf{F}_m(W_2)$ are minimal film-mass representives of the films. Construct the product representative $T \in \mathbf{F}_m(W)$ by installing T_1 in the sheets $(i0, j0)$ and T_2 in the sheets $(0\alpha, 0\beta)$. T then projects down to give the film in Y. When the supports are projected to the base manifold, one gets the union of the supports of Y_1 and Y_2, hence a superposition of films. If ω_1 and ω_2 are calibrations, then a calibration in the product is $\omega_{i\alpha} = \omega_{1i} + \omega_{2\alpha}$. Hence the product film is indeed minimal, at least in the real case. Is it in the integral case also?

§12.6. Different covering spaces. Different covering spaces may give different films, as shown in several of the examples. Can one determine when different covering spaces give different films? The possibility of using "invisible wires" further complicates matters. Throughout this section, we will assume a base space M and a reference surface S on a distinguished sheet A of each covering space. The reference surface S on a covering space Y is assumed to be the lift (to one sheet locally) of a reference surface S_0 in M.

Covering spaces (with invisible wires) form a directed set with respect to the covering relation. For the upper bound of two covering spaces Y_1 and Y_2 one could construct a common covering space Y_{12} in a similar manner to the pair covering space construction:

$$Y_{12} = \{(y_1, y_2) \in Y_1 \times Y_2 : p_1(y_1) = p_2(y_2)\},$$

where p_1 and p_2 are the respective projections to M. If S_1 is the lift of S_0 to sheet i of Y_1 and S_2 is the lift to sheet j of Y_2, then S_{12} is the lift of S_0 to sheet ij of Y_{12}.

Proposition 12.1. *Suppose*

  (1) Y_1 *and* Y_2 *are covering spaces of* M,
  (2) Y_2 *is a cover of* Y_1 *with projection* $\tau : Y_2 \to Y_1$,
  (3) *the reference surfaces are consistent,* $\tau S_2 = \tau S_1$,
  (4) *there is a projection* $\sigma : W_2 \to W_1$ *of the pair covering spaces that is consistent with* τ.

Then for either (IMIN) *or* (RMIN) *the optimal film in* Y_2 *has at least as much film-mass as that for* Y_1.

Proof. Let (Q_2, R_2, T_2) be an optimal film in Y_2. Then the hypotheses are sufficient to imply $(\tau_\# Q_2, \tau_\# R_2, \sigma_\# T_2)$ is a feasible solution in Y_1, and $\mathbf{M}(\sigma_\# T_2) \leq \mathbf{M}(T_2)$.
□

Remarks. If Y_2 covers Y_1, then it is not necessarily true that the film of Y_1 is contained in the film of Y_2.

There is also the question of whether the minimal film in the joined covering space Y_{12} stays away from the closed boundary. Perhaps one should join only covering spaces whose closed boundaries have positive mean curvature or some other property that guarantees the boundary does not become an obstacle to the film.

Proposition 12.2. *With the same hypotheses as Proposition 12.1, suppose the reference current* S_1 *has a unique lift to* S_2. *Suppose that* (Q_1, R_1, T_1) *is an optimal solution in* Y_1 *such that* $spt(Q_1)$ *is simply connected. Then the solution lifts to an optimal solution in* Y_2.

Proof. By simply connectedness and the hypothesis on S_1, the current Q_1 lifts uniquely to a current Q_2 of Y_2, R_1 lifts to $R_2 = \partial Q_2 + S_2$, and T_1 lifts to T_2. Hence (Q_2, R_2, T_2) is a feasible film, and $\mathbf{M}^\dagger(R_1) = \mathbf{M}^\dagger(R_2)$, so it follows from proposition 12.1 that it is a minimal film.   □

Hence, as soon as one gets a film that has a simply connected complement, one knows one has the maximal film for full covering spaces of that M. However, there are boundaries that have several films of different areas with simply connected complements. For example, the double catenoid rings bound a surface of the type of a disk that consists of two disks in one set of rings joined by a strip between the other pair of rings. To get these different films, one needs to choose different M's.

Taking Y as the universal cover of M does not guarantee a simply connected $spt(U)$, as the double catenoid without disks shows (example 7.6).

The universal covering space may not give the largest film on that boundary, as the double catenoids (examples 7.6-8) show. Is there a "universal covering space" (with invisible wires) that could be guaranteed to give the largest film? The case of the octahedral frame (example 8.6) with five films with connected complement is a good example to contemplate here.

13. RELATION TO SOME OTHER SOAP FILM MODELS

§**13.1. Spanning.**   As mentioned in the introduction, many soap film models require the film to "span" the boundary. It may be phrased as requiring the boundary to be the oriented boundary of the film in a homological sense, or requiring the

film to annihilate the homology of the boundary. The covering space model instead focuses on the complement of the boundary. The film is a boundary itself, instead of being a boundary. The spanning requirements involve the fundamental group of the complement of the boundary by way of covering spaces (although invisible wires may prevent formulation in terms of the original boundary alone).

§13.2. **Classical calibration.** Classical calibration problems are phrased in terms of maximizing flux through a surface. This can be translated into a covering space scheme through the method described in sedction 12.2, although in these cases it is probably simpler to stick to the classical scheme.

§13.3. **Paired calibrations.** The covering space calibration method is a generalization of the paired calibrations of [LM] and [B1]. Paired calibrations handle interfaces between regions. Each region i has a reference current S_i and a flow ω_i. The constraint is that any two flows must have magnitude difference at most 1. Clearly this corresponds to the covering space calibration where the covering space consists of one copy of M for each region, and each S_i is a component of the overall S that winds up on sheet i.

§13.4. **Twisted calibrations.** Murdoch's twisted calibrations [MU] may be seen as a special case of covering space calibrations. Murdoch considers a possibly unoriented submanifold N of a Riemannian manifold M. There is a line bundle L on M that orients N in the sense that integration of L-valued forms can be done over N. A twisted calibration is a form with coefficients in L that reduces to the volume form on N. Clearly the line bundle is equivalent to a double cover of M since lines have two possible orientations. The twisted forms then correspond to forms on the double cover with opposite values on the two sheets.

§13.5. **Least fundamental domain.** For a smooth Riemannian three-dimensional manifold, Choe [CJ] finds a fundamental domain with least boundary area in the universal covering space. Such a fundamental domain is often the result of my covering space model, except that I am working in a manifold with boundary. If the manifold has no boundary, then the fundamental domain boundary is homologous to zero. Choe works in the rectifiable set model, and his techniques are limited to three dimensions.

14. CONCLUSION

The covering space model for soap films carries the promise of a "soap film technology" in which there are standard algorithms for answering the fundamental questions about films. For a given boundary covering spaces may be constructed, and films may be found by maximizing flow. Calibration methods can prove minimality and uniqueness, at least for real currents. Lots of other features can be plugged into the basic scheme: volume constraints, gravitational energy, free boundary contact angles, variable surface tension. But those will be the subject of a future paper.

I close with a list of conjectures and questions, some mentioned earlier:

(1) Are there any soap film phenomena which the covering space model cannot handle? Readers are invited to send challenging films to the author.
(2) Discover the proper barrier theory for handling triple junctions and other soap film singularities. (§11.2)

(3) If the reference surface S is integral, what can be said about the minimal surface? (§10.2)

(4) Find all possible types of singularities of integral soap films when the pair covering space W is not the full pair covering space.

(5) Can every stable $(\mathbf{M}, 0, \delta)$-minimal set be represented with the covering space model? Find an algorithm to do so. (§12.1)

(6) Develop a test for telling whether two different covering spaces will give the same soap film. (§12.6)

(7) For a given boundary, generate all covering spaces that can give different films. The possibility of invisible wires (§5.3) makes this tricky. Find all films on a boundary. For example, does the octahedron have more than the five of figure 20?

(8) Of all the films on a boundary (e.g. on an octahedral frame), find the one of maximal area and prove it maximal.

(9) Calibrate the supposed minimal octahedral film, or find a real flat chain with smaller mass. (Example 8.8)

(10) Is there an a priori bound on the number of sheets necessary in the covering space for a given film, possibly depending on how invisible wires are used? (Example 8.8)

(11) Design an efficient computer algorithm to find the maximum flow for a given covering space, and finding the corresponding soap film. (§11.1) A beginning has been made in [B4].

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