Minimal cones on hypercubes

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Abstract. It is shown that in dimension greater than 4, the minimal area hypersurface separating the faces of a hypercube is the cone over the edges of the hypercube. This constrasts with the cases of two and three dimensions, where the cone is not minimal. For example, a soap film on a cubical frame has a small rounded square in the center. In dimensions over 6, the cone is minimal even if the area separating opposite faces is given zero weight. The proof uses the maximal flow problem that is dual to the minimal surface problem.

Math subject classification:
Primary: 49Q05 Minimal surfaces
Secondary: 49Q20 Variational problems in a geometric measure-theoretic setting
49N15 Duality theory

Originally published in The Journal of Geometric Analysis, vol. 1, number 4, 1991, pp. 329–338.

1. Introduction.

Part of knowing the structure of minimal surfaces is knowing what kinds of singularities are possible. By repeated magnification and a compactness argument, a singular point can be shown to have a limiting shape, or *tangent cone*. A *cone* is an object that is invariant under homothety from the origin. To be a candidate for a singularity type, a conical surface must be minimizing in whatever class of surfaces is being considered.

The first nontrivial minimal tangent cone proven for hypersurfaces was the cone over $S^3 \times S^3$ in $\mathbf{R}^8$ in [BDG]. Jean Taylor [TJ] proved that the only two singularities possible in soap films in $\mathbf{R}^3$ have tangent cones that are three planes meeting at $120°$ or six planes forming a cone over the edges of a regular tetrahedron. In particular, a soap film on a cubical wire frame does not form a cone of twelve triangles meeting at the center; rather a curved central square forms. Lawlor and Morgan [LM] show that the cone over the $(N-2)$-dimensional skeleton of a regular simplex in $\mathbf{R}^N$ is area minimizing among soap-film-like hypersurfaces. This paper shows that for the corresponding problem for a hypercube in dimension four or greater the cone is area minimizing. Thus the set of singularities in higher dimension is much richer than just cones over regular simplices.

Many different soap films can form on a cubical frame. The ones considered here are those that span the frame in the sense of separating all the faces of the cube from each other. These surfaces can be regarded as the interfaces between several immiscible fluids, one for each face. The problem can be generalized to permit different surface tensions between different pairs of fluids. This paper considers the following problem:

1. The domain is a hypercube in $\mathbf{R}^N$ centered at the origin with side length 2.
2. The hypercube is partitioned into $2N$ regions with each face belonging to a corresponding region.
3. The area of the boundary between regions of adjacent faces has weight 1, and the weight of the boundary between regions of opposite faces is a nonnegative value T. The weights correspond to the surface tensions between fluids.
4. The goal is to minimize the weighted boundary area.

Note that there are no volume constraints on the regions. In the fluid interpretation, this means that all the regions are connected to external resevoirs at equal pressure.

The results may be summarized in terms of the critical tension T_c, defined as the least value of T for which the cone is minimizing:

$$\begin{aligned} N = 2,3: &\qquad T_c = \sqrt{2} \\ N = 4: &\qquad 0.545 < T_c < 0.94 \\ N = 5: &\qquad T_c < 0.5 \\ N = 6: &\qquad T_c < 0.13 \\ N \geq 7: &\qquad T_c = 0 \end{aligned}$$

The upper bounds for $N = 4, 5, 6$ are rather crude, but suffice to show the cone is minimizing for plain area for $N \geq 4$.

2. Notation and geometry.

N is the dimension of the ambient space, $\mathbf{R}^N$.

The hypercube is the region $[-1,1]^N$.

The coordinates will be grouped into an $(N-2)$-component vector $\mathbf{x}$, and coordinates y and z. When attention is focused on a pair of opposite faces, they will be the $z=1$ and $z=-1$ faces. The generic third face will be the $y=1$ face. When there is a central square, it will be in the $z=0$ plane.

m-dimensional surfaces will be represented as normal m-currents [FH, 4.1.7]. Currents are the dual space of differential forms, and normal currents are those representable by integration, i.e. if ω is an m-form and S is a normal m-current, then there is a Radon measure $||S||$ and a $||S||$-measurable m-vectorfield $\mathbf{u}$ such that $||\mathbf{u}||=1$ almost everywhere with respect to $||S||$ and

$$\int_S \omega = \int < \omega, \mathbf{u} > d||S||.$$

The measure $||S||$ is the area measure of the current. All currents hereafter are assumed to be normal currents. The forms ω dual to normal currents are those for which both ω and $d\omega$ are representable by locally bounded Lebesgue measurable covectorfields. For convenience, $(N-1)$-forms will be represented as their dual vectorfields.

In a problem with M regions to be separated, a *face tuple* F is taken to be a set of $(N-1)$-dimensional currents $F_1, \ldots, F_M$. A *dividing surface* H for F will be a set of currents $H_{ij} = -H_{ji}$ for $1 \le i \ne j \le M$ such that

$$F_i + \sum_j H_{ij} = \partial B_i \qquad \text{for some N-current } B_i \text{ for each } i.$$

For a dividing surface H to exist, it is sufficient that $\sum_i F_i$ be the boundary of some region. The F_i will be the faces of a hypercube in this paper. Signed coordinate subscripts may indicate the face; for example F_z is the $z=1$ face and F_{-z} is the $z=-1$ face.

The unit vector from the origin to the center of face F_i is $\mathbf{e}_i$.

The *face-cone* of a face is the cone whose base is the face and whose vertex is the origin. The *cone over the hypercube* is the dividing surface where H_{ij} is the boundary between the face-cones of F_i and F_j, oriented as positive boundary of the face-cone of F_i.

Each face tuple will have associated with it a set of real numbers (interface energies) $a_{ij} = a_{ji} \ge 0$ for $1 \le i \ne j \le M$. The **mass** of a dividing surface H is the sum of the weighted areas of the interface components:

$$mass(H) = \sum_{i<j} \int a_{ij} d||H_{ij}||.$$

A *flow* $\mathbf{V}$ for F is a set of divergenceless vectorfields $\mathbf{v}_1, \ldots, \mathbf{v}_M$ such that

$$|\mathbf{v}_i - \mathbf{v}_j| \le a_{ij} \qquad \text{pointwise in } \mathbf{R}^N.$$

Here *divergenceless* means $\int_{\partial B} \mathbf{v}_i = 0$ for any N-current B. The *flux* of $\mathbf{V}$ through F is

$$flux(F, \mathbf{V}) = \sum_i \int_{F_i} \mathbf{v}_i.$$

Signed coordinate subscripts may indicate the face a vectorfield is associated with; for example, $\mathbf{v}_y$ is associated with F_y.

A formulation that would work equally well would be to require for a dividing surface that

$$\partial F_i + \sum_j \partial H_{ij} = 0 \qquad \text{for each } i,$$

and that the vectorfields of a flow correspond to exact forms.

3. Maximal flow.

Upper bounds on area can be found by constructing explicit surfaces. The dual problem to finding a minimal surface spanning a boundary is to find a maximal flow through the boundary. This is a continuous version of the duality between maximal flows and minimal cuts in network theory. Full duality is proved in a limited context by Federer [FH2] and more widely in [BK1]. We will not need the full duality, but just an easily proved part of it, so we may use explicit flows to find lower bounds on area. The partial duality is expressed in the following theorem, due to Lawlor and Morgan [LM] in the case of constant vectorfields.

Theorem 3.1. *Suppose F is a face tuple. Then*

$$\inf\{mass(H) | H \text{ is a dividing surface for } F\}$$

$$\geq \sup\{flux(F, \mathbf{V}) | \mathbf{V} \text{ is a flow for } F\}.$$

Proof. Suppose F is a face tuple, $\mathbf{V}$ is a flow for F, and H is a dividing surface for F. Then

$$mass(H) = \sum_{i<j} \int a_{ij} d||H_{ij}|| \geq \sum_{i<j} \int |\mathbf{v}_j - \mathbf{v}_i| d||H_{ij}|| \geq \sum_{i<j} \int_{H_{ij}} \mathbf{v}_j - \mathbf{v}_i$$

$$= \sum_{i \neq j} \int_{H_{ij}} -\mathbf{v}_i = \sum_i \int_{F_i} \mathbf{v}_i = flux(F, \mathbf{V}).$$

∎

Remark. In the case that a flow $\mathbf{V}$ is maximal (for example, when its flux is the same as the mass of a known dividing surface), then each H_{ij} of an absolutely minimizing dividing surface must have its support in the critical set K_{ij} where

$$K_{ij} = \{x \in \mathbf{R}^n : |\mathbf{v}_i - \mathbf{v}_j| = a_{ij}\}.$$

In cases where there are multiple absolutely minimizing dividing surfaces (as in section 5), the critical sets for each maximal flow must contain the supports of all those dividing surfaces.

4. Cones are minimizing for $T \geq \sqrt{2}$.

As a simple example of the use of Theorem 3.1, one can easily prove:

Theorem 4.1. *If $N \geq 2$ and $T \geq \sqrt{2}$ then the cone over the hypercube is absolutely minimizing.*

Proof. The cone has area $2^{N-1}\sqrt{2}N$. The vectorfields $\mathbf{v}_i = \mathbf{e}_i/\sqrt{2}$ form a flow for F and have total flux $2^{N-1}\sqrt{2}N$. Therefore the cone is absolutely minimizing. ∎

5. Dimension 2, $T < \sqrt{2}$.

The minimal surface H here is well known to consist of five line segments, four from the corners meeting the ends of a segment through the origin. There are two possible orientations of the surface, with the central segment horizontal or vertical. Suppose it to be horizontal. Then the endpoints are at

$$\left(-1 + \frac{T}{\sqrt{4-T^2}}, 0\right) \qquad \text{and} \qquad \left(1 - \frac{T}{\sqrt{4-T^2}}, 0\right).$$

The total mass (weighting the central segment by T, recall) is $2T + 2\sqrt{4-T^2}$.

A maximal flow can be defined as follows:

$$\mathbf{v}_z = (0, T/2), \qquad \mathbf{v}_{-z} = (0, -T/2),$$

$$\mathbf{v}_y = \begin{cases} (\sqrt{1-T^2/4}, 0) & \text{for } y \geq |z|, \\ (T/2, -\sqrt{1-T^2/4} + T/2) & \text{for } z > 0,\ -z < y < z, \\ (T/2, \sqrt{1-T^2/4} - T/2) & \text{for } z < 0,\ -|z| < y < |z|, \\ (T - \sqrt{1-T^2/4}, 0) & \text{for } y \leq |z|. \end{cases}$$

Define $\mathbf{v}_{-y}$ to be the reflection of $\mathbf{v}_y$ in the z axis. Note that $\mathbf{v}_y$ changes direction when it hits the diagonals of the square in such a way as to conserve flux, so it is divergenceless. The total flux is $2T + 2\sqrt{4-T^2}$, so the configuration H is absolutely minimizing.

The existence of a maximal flow does not prove uniqueness of the minimal surface, as the existence of two orientations of H here clearly shows. But for equality to hold in Theorem 3.1, the normals of H must be parallel to $\mathbf{v}_j - \mathbf{v}_i$ where $|\mathbf{v}_j - \mathbf{v}_i| = a_{ij}$. The flow given here restricts the normals to just a few directions, whence it can be proved the two orientations are the only solutions.

6. Dimension 3, $T < \sqrt{2}$.

For $T = 1$, one gets experimentally the soap film with a rounded central square. This surface has eight curved sections with unknown equations, so finding a flow to prove it optimal would be extremely difficult. By calculation with my Surface Evolver program [BK2], it has area approximately 16.598, which is less than the cone area 16.971. So I cannot prove it absolutely minimizing, but it does show the cone is not. The question is, does raising the surface tension in the central square make it disappear for some $T < \sqrt{2}$? The answer is no.

Theorem 6.1. *For $T < \sqrt{2}$ there exists a spanning surface whose area is less than that of the cone.*

Proof. The cone has mass $12\sqrt{2}$. The surface H constructed will have the general shape of the cubical soap film, but with the curved faces being cylinders with generator $z = y + c\log y$, where $c > 0$. Note that $z = 1$ at $y = 1$, so the surface fits the cubical frame. The surface hits the central square at $0 = y_0 + c\log y_0$. The total mass of the surface is

$$mass(H) = 16\int_{y_0}^{1} \sqrt{1+z'^2}\,y\,dy + 8\int_{y_0}^{1} \sqrt{2}z\,dy + 4Ty_0^2$$

$$= 16\sqrt{2}\int_{y_0}^{1} \sqrt{y^2 + cy + \frac{c^2}{2}}dy + 8\sqrt{2}\int_{y_0}^{1} y + c\log y\,dy + 4Ty_0^2$$

$$\leq 16\sqrt{2}\int_{y_0}^{1} y + \frac{c}{2} + \frac{c^2}{8y}dy + 8\sqrt{2}\int_{y_0}^{1} y + c\log y\,dy + 4Ty_0^2$$

$$= 16\sqrt{2}\left[\frac{1-y_0^2}{2} + \frac{c(1-y_0)}{2} - \frac{c^2}{8}\log y_0\right]$$

$$+ 8\sqrt{2}\left[\frac{1-y_0^2}{2} - c - cy_0\log y_0 + cy_0\right] + 4Ty_0^2$$

$$= 12\sqrt{2}(1-y_0^2) - 2\sqrt{2}c^2\log y_0 - 8\sqrt{2}cy_0\log y_0 + 4Ty_0^2$$

$$= 12\sqrt{2}(1-y_0^2) - 2\sqrt{2}\frac{y_0^2}{\log y_0} + 8\sqrt{2}y_0^2 + 4Ty_0^2.$$

Thus we have mass less than the cone for

$$-4\sqrt{2}y_0^2 - 2\sqrt{2}\frac{y_0^2}{\log y_0} + 4Ty_0^2 < 0$$

or

$$T < \sqrt{2} + \frac{1}{\sqrt{2}\log y_0}.$$

We can pick y_0 as small as we please and still get $c > 0$, thus getting comparison surfaces for all $T < \sqrt{2}$. ∎

These surfaces make no attempt to have zero mean curvature, so they are probably not good guides to the size of the central square. A much better approximation to the actual optimal surface would probably come from linearizing the minimal surface equation

$$(1+z_y^2)z_{xx} - 2z_xz_yz_{xy} + (1+z_x^2)z_{yy} = 0$$

for $z = y + h(x,y)$ for small h. This gives

$$2h_{xx} + h_{yy} = 0.$$

A solution could be

$$z = y + c\log(x^2 + 2y^2) - c\log(x^2 + 2(2-y)^2).$$

The last term is there to make $z = 1$ at $y = 1$ and will be neglected hereafter. An accurate area for this surface is much tougher to calculate than the one used above. But, assuming it is a good approximation to the optimum, we can find T from the contact angle at $z = 0$. Take $x = 0$. Then $z \approx y + c\log 2y^2$, $dz/dy \approx 1 + 2c/y$ and $0 \approx y_0 + c\log(2y^2)$. So $z'(y_0) \approx 1 - 1/\log(2y_0^2)$. Further, from a balance of forces argument,

$$\frac{T}{2} = \frac{1}{\sqrt{1+z'^2}},$$

so

$$T \approx \sqrt{2}\left(1 + \frac{1}{\log(2y_0^2)}\right),$$

so

$$y_0 \approx \frac{1}{\sqrt{2}}\exp\left(\frac{-1}{2\sqrt{2}(\sqrt{2}-T)}\right).$$

None of this proves that the optimum surface actually has a central square; it just proves that the cone is not minimizing.

7. Dimension 4 comparison surfaces.

The comparison surface H here will be the four dimensional analogue of the cube film with a central "square" of side length $2y_0$, 12 curved cylindrical surfaces meeting the sides of the square (one of positive z and one of negative z on each of its 6 faces) and 12 flat surfaces joining the edges of the cylindrical surfaces. A nice, simple formula for the curve doesn't seem to work, so I apply some calculus of variations to find the optimal cylinder and numerically integrate.

The cylinder generator curve will be $y = y(z)$ for $0 \le z \le 1$. The total mass will be

$$mass(H) = 48\int_0^1 (1+y'^2)^{1/2}y^2 + \frac{1}{\sqrt{2}}(1-y^2)dz + 8Ty_0^3.$$

The Euler equation turns out to be

$$y' = \frac{(y^4 - 2cy^2 - c^2)^{1/2}}{y^2 + c},$$

where c is an arbitrary positive constant. Some numerical results (with T_{max} being the maximum T for which $mass(H) \le mass(cone)$):

c	T_{max}	y_0
0.00000001	0.54490	0.00016650
0.00000010	0.54470	0.00052648
0.00000100	0.54407	0.00166449

0.00001000	0.54208	0.00525974
0.00010000	0.53577	0.01659466
0.00100000	0.51561	0.05210906
0.01000000	0.44970	0.16151862
0.10000000	0.21227	0.49201093

Note that there seems to be an upper bound on T, which we will see in the next section to be real.

8. Flows for dimension ≥ 4.

The flows constructed in this section will show that the cone H over the hypercube is minimizing. The reason cones are minimizing in higher dimensions is that the flow can spread out before running into the flow from the opposite face.

The flow in the face-cone of F_i is $\mathbf{v}_i = \mathbf{e}_i/\sqrt{2}$. The rest of the flow has to be more carefully built. The flow $\mathbf{v}_z$ for the face F_z will be constructed; the flows for the other faces are similar. Of $\mathbf{v}_z$, the portion of the flow through the face-cone of the face F_y will be detailed, with the flows through the other adjacent face-cones being symmetric. In the face-cone of F_y, the coordinates are $(\mathbf{x}, y, z)$, where $0 \leq y \leq 1$, $|x_i| \leq y$, and $|z| \leq y$. To take advantage of radial spreading, the flow at $(\mathbf{x}, y, z)$ will have the form $(\alpha\mathbf{x}/y, \alpha, \beta)$, where $\alpha(\mathbf{s}, t)$ and $\beta(\mathbf{s}, t)$ depend on $\mathbf{s} = \mathbf{x}/y$ and $t = z/y$. Hence the flow is invariant under homothety, just as the cone is. Several conditions must be satisfied:

1. The flux through H_{yz} must match on both sides. The normal of H_{yz} is $\mathbf{e}_z - \mathbf{e}_y$, so

$$1/\sqrt{2} = \beta - \alpha \qquad \text{at } z = y.$$

2. The flow is divergenceless. This translates to

$$(N-2)\alpha - t\alpha_t + \beta_t = 0.$$

3. Bounded difference from the flow $\mathbf{V}_y = \mathbf{e}_y/\sqrt{2}$ of face F_y:

$$\alpha^2 s^2 + (1/\sqrt{2} - \alpha)^2 + \beta^2 \leq 1.$$

4. Bounded differences from the flows belonging to the other faces adjacent to both F_z and F_y. This turns out to be a consequence of condition 3.
5. Bounded difference from the flow of the opposite face F_{-y}:

$$|\mathbf{v}_z - \mathbf{v}_{-y}| \leq 1.$$

6. Bounded difference from the flow of the opposite face F_{-z} in the face-cone of F_{-z}:

$$|\mathbf{v}_z - \mathbf{v}_{-z}| \leq T.$$

For fixed s^2, we can numerically integrate conditions 2 and 3 (taken as equality) from the initial condition $\alpha = 0$, $\beta = 1/\sqrt{2}$ at $t = 1$ in the direction of decreasing t until the flow is radial at some t_c, $\beta/\alpha = t_c$. At this point, there is no flux across the $t = t_c$ plane, and the vectorfield can be cut off.

Note that condition 3 is most stringent at the maximum value of s^2, which is $N-2$.

For $N = 4$, the flow becomes radial at $t_c \approx -.81$ for $s^2 = 2$. The difference between $\mathbf{v}_z$ and $\mathbf{v}_{-z}$ has its maximum value of 0.9333 at $z = 0$. Hence condition 6 at $z = 0$ is satisfied if $T > .9333$. Take the flow to be null for $t > T_c$ and in the face cone of F_{-z}. This makes condition 5 easily satisfied.

For $N = 5$, the flow becomes radial at $t_c \approx -.297$ for $s^2 = 3$. The difference between $\mathbf{v}_z$ and $\mathbf{v}_{-z}$ has its maximum value of 0.4974 at $z = 0$. Hence condition 6 at $z = 0$ is satisfied if $T > .4974$. If $T \geq 1/\sqrt{2}$, the flow for $t < t_c$ can be left as null. Otherwise, to satisfy condition 6, some flow of $\mathbf{v}_z$ will have to be put through the face-cone of F_{-z}. Make this part of the flow $(-\sqrt{2}+T)\mathbf{e}_z$. Continue it into the face-cone of F_y in the same manner as before, starting at $t = -1$. By numerical integration, this flow goes radial well before t_c for $T > T_c = 0.4974$. Hence the cone is minimal for $T > 0.4974$.

For $N = 6$, the situation is like that for $n = 5$, with $t_c \approx -0.059$ and $T_c \approx 0.12$.

For $N = 7, 8$, the numerical integration shows that the flow becomes radial for $t_c > 0$, so the flow does not reach the $z = 0$ plane. Adjoin to $\mathbf{v}_z$ its reflection in the $z = 0$ plane. Now $\mathbf{v}_z = \mathbf{v}_{-z}$, so condition 6 is satisfied for $T = 0$, and condition 5 becomes equivalent to condition 3. Hence the cone is absolutely minimizing for $T \geq 0$.

For $N \geq 9$, one can take

$$\alpha = \frac{\sqrt{2}(1-t)}{n+1},$$
$$\beta = \frac{(n-1)(1-t)^2}{\sqrt{2}(n+1)} + \frac{\sqrt{2}(1-t)}{(n+1)} - \frac{1}{\sqrt{2}}$$

for $0 < t < 1$ and the reflection for $-1 < t < 0$. These satisfy all the conditions and $\beta = 0$ at $t = 0$, so $T_c = 0$.

These flows are relatively crude, and further work will improve the upper bounds on T_c. Almost certainly $T_c = 0$ for $N = 6$, and possibly for $N = 5$. But these results are sufficient to show that the cone over the hypercube is absolutely minimizing for plain area ($T = 1$) for $N \geq 4$. Further, note that condition 3 need be equality only at $z = y$, which shows that the cones are the unique absolutely minimizing dividing surfaces.

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