The Opaque Cube Problem

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1. Introduction.

The Opaque Square Problem has been floating around a long time:

What is the shortest length fence that can block any line of sight across a square plot of ground?

The best known solution is shown in figure 1. It has straight fences from three corners meeting at a point at angles of $2\pi/3$ plus a fence from the fourth corner to the center. It has not been proved that this is in fact the best possible. For more on opaque plane regions, including opaque circles and polygons, see [**5**].

Martin Gardner [**6**] has raised the Opaque Cube Problem:

What is the least area surface that can block all lines of sight through a cube?

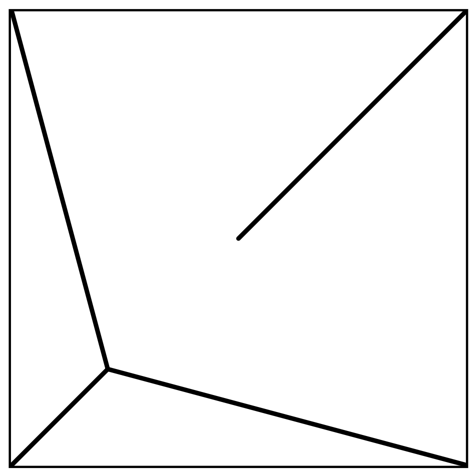

Figure 1. The best known opaque square solution.

In any dimension, one has the Opaque Region Problem:

What is the least measure hypersurface that intersects all lines that pass through a given region?

One can pose two versions of each problem: the restricted version, which permits fences only inside the region, and the unrestricted version, which also permits fences outside. Despite the simplicity of the statement of the problem, practically nothing has been proved for any region. No fence has been proved optimal for any region in dimension 2 or higher that does not lie in a hyperplane. Even the opaque equilateral triangle is unproved.

I will present a possible solution to the Opaque Cube Problem, and I will use the Opaque Sphere Problem to suggest that the optimal surface may not exist. To get a solution, it may be necessary to widen the type of object considered as a fence, to include varifolds, for example. In that case, the notion of "opaqueness" needs clarification. I use the term *fence* to refer to the object making the region opaque in order not to prejudge what type of object is proper.

2. The Opaque Cube Problem.

Let the region to be made opaque be a unit cube. An obvious way to make it opaque is with twelve triangles from the edges to the center, as shown in figure 2. It has area $3\sqrt{2} \approx 4.2426$. However, this cannot be the best because the central vertex is not one of the types allowed in minimal surfaces. The only types of singularities found in the interiors of minimal surfaces are three surfaces meeting along a curve at angles of 120° or six surfaces meeting at a point with tetrahedral angles [**9**].

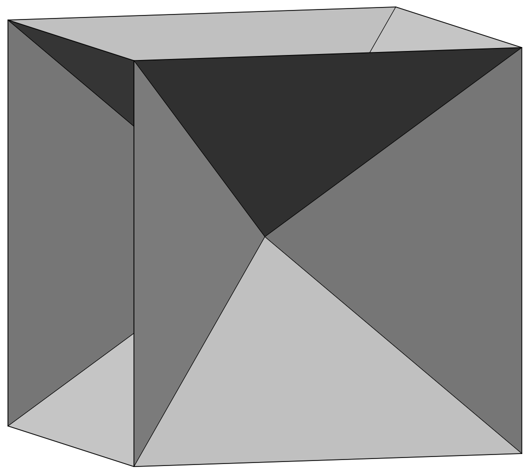

Figure 2. An opaque cube solution with twelve triangles meeting at the center. The area is $3\sqrt{2} \approx 4.2426$.

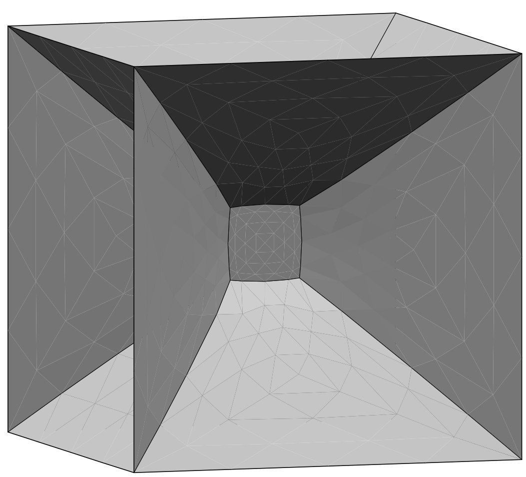

Figure 3. The soap film that forms on a cubical frame, with a central rounded square. Area ≈ 4.2396.

If a cubical wire frame is dipped in soap solution, then the surface that forms is shown in figure 3. The central vertex of figure 2 has been replaced by a rounded square, and all the singularities are of the proper type. It has an area of approximately 4.2398. (All the areas cited hereafter in this section were calculated with my program called the Surface Evolver [**2**].) But this is not the best possible solution.

A better solution (see figure 4) can be constructed as a three dimensional version of the Opaque Square solution. The best way to visualize this surface is to begin with a non-optimal fence made up of flat planes and then imagine it shrinking like a soap film to the final state shown in figure 4. Begin with a cubical frame. Add the top and bottom faces of the cube. Add four vertical rectangles between the top and bottom faces so that their horizontal cross-section is the opaque square solution. The central gap in the opaque square solution becomes a tunnel through the cube, but of course there is no line of sight through the tunnel. When this initial configuration is run through the Surface Evolver to minimize area, the top face gets pulled down and the bottom face gets pulled up. The area is approximately 4.2342. There is still a tunnel from the front face, through the middle, and out to the right side, but the vertical edge of the surface in the middle is constrained to stay on the vertical centerline, so one cannot see all the way through the tunnel. This surface could be made as a real soap film if a central vertical wire were added to a cubical frame, but it might be tricky to convince the soap film to take up this particular topology.

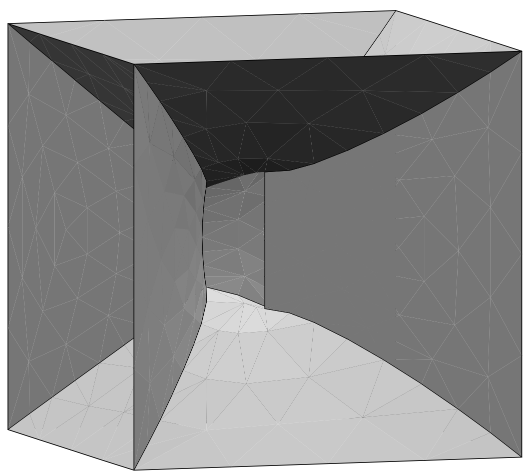

Figure 4. A better opaque cube solution. A horizontal slice through the middle looks like the opaque square solution. Area ≈ 4.2342.

My best solution is shown in figure 5. It is similar to figure 4, except that twofold symmetry has been replaced by threefold symmetry. The area is approximately 4.2324. There are three entrances to the central tunnel, from the front, from the bottom, and from the right. Another way to describe the topology is to start with a soapfilm on a cubical frame with a cubical bubble in the middle, and then remove three adjacent faces of the bubble. A soap film version would need three wires coming out from the center at right angles to hold the edges of the surface.

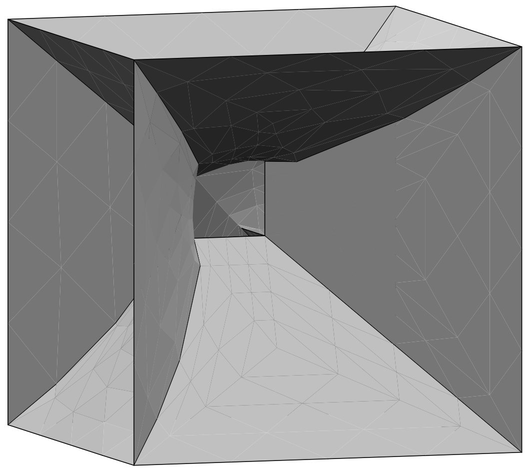

Figure 5. The best known opaque cube solution. It is like figure 4, but with threefold symmetry in place of twofold. Area ≈ 4.2324.

A videotape showing these shapes is available in [**3**].

Martin Gardner [**7**] is offering a $50 prize for "the best improvement" on figure 5.

3. The Opaque Sphere Problem.

This section will construct a sequence of fences for a unit sphere that converges to a set of larger area than the limit of the areas. The problem will be the unrestricted version, permitting fences outside the sphere. The first fence F_1 consists of the lower hemisphere plus a cylinder around the upper hemisphere, as shown in figure 6. (The top of the cylinder is not included.) The area is $A_1 = 4\pi$, which is the same as the area of the sphere. Each successive fence F_{n+1} is formed by slicing each cylinder of F_n in half horizontally and shrinking the top half until it hits the sphere. This sequence was first found by R. Laver, as cited in [**5**]. The areas A_n form a strictly decreasing sequence, and

$$A_\infty = \lim_{n\to\infty} A_n = 2\pi + \int_0^1 2\pi\sqrt{1-z^2}\,dz = 2\pi + \pi^2/2.$$

Note, however, that the limiting point set is the surface of the sphere, which has a larger area than A_∞. This suggests that the least area hypersurface of the Opaque Region Problem may fail to exist. Can the Opaque Region Problem be reformulated so that a solution can always be proved to exist in some sense?

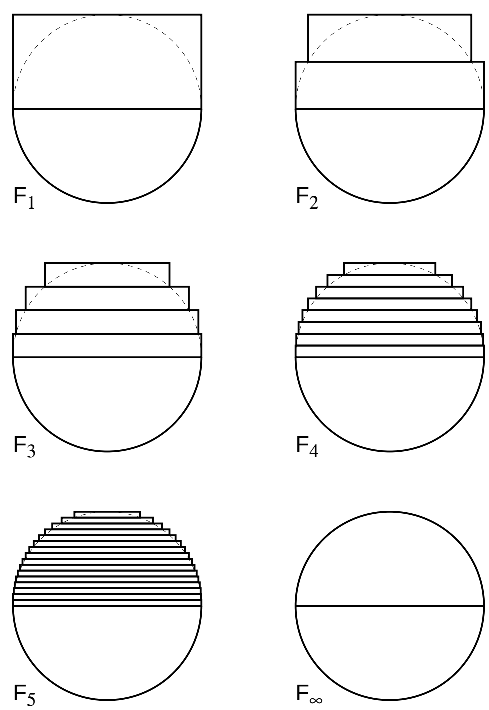

Figure 6. A sequence of varifold solutions, with limiting varifold F_∞.

4. A varifold formulation.

A standard method of solving minimization problems is to find a minimizing sequence of objects, use compactness to guarantee the existence of a limit object, and show the limit object is a valid solution and absolutely minimizes the objective function. If the opaque sphere sequence F_n is in fact a minimizing sequence, then this strategy fails if the objects are point sets. This example shows that we need to reconsider the type of object that a fence is.

To make the compactness argument succeed, fences need to be from a topological space in which area is lower semicontinuous and the limit of an opaque minimizing sequence is also opaque. Clearly the tangent planes of the F_n must be taken into account in the limit. Varifolds provide a setting in which area and tangent planes behave properly in the limit. A k-dimensional **varifold** in $\mathbf{R}^m$ is a measure on $\mathbf{R}^m \times G_k\mathbf{R}^m$, where $G_k\mathbf{R}^m$ is the Grassmannian manifold of unoriented k-planes through a point (see [**1**],[**8** p. 109]). In other words, the measure is on planes at points, not just points. The varifold area (or **mass**) is the total measure of $\mathbf{R}^m \times G_k\mathbf{R}^m$, and the space of varifolds is compact with area being lower semicontinuous. A smooth manifold naturally corresponds to a varifold in which the measure is on the geometric tangent plane at each point.

If the sphere fences F_n are regarded as varifolds, then the limit varifold F_∞ exists and behaves as desired. The upper hemisphere of F_∞ has all of its measure on vertical planes, a sort of infinitesimal venetian blind effect, and the area of F_∞ is $A_\infty = 2\pi + \pi^2/2$.

There remains to be stated a definition of opaqueness for varifolds. I propose the following. First, define a point P to be a **point of opacity** for a line L if the projection of the varifold in any neighborhood of P on the perpendicular hyperspace of L has at least unit density at the projection of P. Second, say that a varifold makes a region **opaque** or is a **fence** if almost every line that intersects the region has a point of opacity on it. "Almost every" is understood in the measure theoretic sense on the manifold of lines.

This definition says "almost every" because in some solutions that we want to keep (such as the opaque cube solutions with tunnels) there are lines that graze the edges of fences. The projected density locally along these lines is only 1/2. These points cannot be counted as points of opacity, or else the density all over could be cut down. Another alternative to "almost every" would be to say that a line is blocked if some arbitrarily near line has a point of opacity. But then one could block all lines with an arbitrarily thin but dense dust of tiny varifolds, which again thwarts our purpose.

The limit sphere varifold F_∞ is opaque in this sense. In particular, every nonhorizontal line that intersects just the upper hemisphere has a point of opacity at its lower intersection with the hemisphere surface. Here the points of opacity of the limit are the limits of the points of opacity of the F_n since the support of the limit F_∞ is a manifold.

It is not clear in general that the limit of a minimizing sequence of opaque varifolds must be an opaque varifold. It is conceivable that the limit varifold may be smeared out so that there would be no points of opacity. On the other hand, perhaps the constraint of being a minimizing sequence is strong enough to force the limit to behave properly.

5. Open Problems.

I conclude with a list of open problems and topics for research:

1. Prove the opaque square solution in figure 1 is optimal.
2. Is the limit varifold of a minimizing sequence of opaque varifolds opaque?
3. Find an example where the solution is provably a varifold, or some other non-manifold.
4. Find a plane region whose optimal fence is plausibly a varifold, or prove that varifolds are not needed in two dimensions.
5. Find an example where restriction of the fence to the region is provably significant.
6. Find the maximum opaque volume for a given area. Is it a hemisphere?
7. For dimension four or greater, the cone over a hypercube (analogous to figure 2) does minimize area [**4**]. Is it also the optimal fence?

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