SOME NEW VALUES OF SYLVESTER'S FUNCTION FOR n NONCOLLINEAR POINTS

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INTRODUCTION. In 1893 J. J. Sylvester conjectured that given a finite number of noncollinear points in the plane and all the lines determined by them, there must be at least one line containing exactly two of the points. The conjecture was first proved by Gallai in 1933. The problem then became: *for all configurations of* n *noncollinear points in the plane and the lines determined by them, what is the minimum number of lines containing exactly two of the points?* For $n > 13$ this is Research Question 10 in this journal (see Volume 1, #2, 1969, page 133).

A line containing exactly two of the n points is called an *ordinary line*. The number of lines containing exactly i of the points is denoted by t_i. The least number of ordinary lines possible for any configuration of n points is denoted by $t_2(n)$. We call this *Sylvester's function*. Results are derived for points on the real projective plane because of the ease of construction of certain configurations and because of the duality between points and lines. Because we are concerned with a finite number of points, results in the projective plane apply also to the Euclidean plane, and conversely.

Gallai in 1933 proved $t_2(n) \geqq 1$. Dirac showed $t_2(n) \geqq 3$ in 1951. In 1958 Kelly and Moser [2] proved $t_2(n) \geqq (3/7)n$. This is the best lower bound presently known. Motzkin and Böröczky have constructed examples giving upper bounds for $t_2(n)$. If $n = 2k$, then $t_2 = k$ can be achieved by constructing a regular k-gon and the k points at infinity determined by lines through pairs of vertices. If $n = 4k + 1$, then $t_2 = 3k$ can be achieved by adding one point at the center of the polygon in the configuration for 4k points. If $n = 4k + 3$, then $t_2 = 3k$ can be achieved by deleting from the $4k + 4$ configuration a point at infinity not determined by an edge of the polygon. The values of $t_2(n)$ presently known are listed below.

n	3	4	5	6	7	8	9	10	11	12	13	14	16	18	22
$t_2(n)$	3	3	4	3	3	4	6	5	6	6	6	7	8	9	11

For proofs and diagrams for $3 \leqq n \leqq 13$, as well as bibliographic references, see Crowe-McKee [1]. The last four values are the subject of this paper.

1. USEFUL FORMULAS. In this section we establish the formulas (1), (10), (12), which are the essential tools for deriving our new values of $t_2(n)$. In a configuration of n points and their connecting lines there are $\binom{n}{2} = [n(n-1)]/2$ pairs of points, each of which must be included in some line. A line containing i of the points includes $\binom{i}{2}$ of the pairs. Thus

$$\binom{n}{2} = \sum_{i=2} \binom{i}{2} t_i \qquad (1)$$

Next consider the dual of a configuration in the projective plane. In the dissection of the plane by this dual, let V_i denote the number of vertices with i edges, F_i the number of regions with i edges. Let V, E, and F denote the total number of vertices, edges, and regions. Then, since we have only even vertices,

$$V = V_4 + V_6 + V_8 + \cdots \tag{2}$$

$$F = F_3 + F_4 + F_5 + \cdots \tag{3}$$

Since each edge has two vertices and each edge belongs to two regions,

$$2E = 4V_4 + 6V_6 + 8V_8 + \cdots \tag{4}$$

$$2E = 3F_3 + 4F_4 + 5F_5 + \cdots \tag{5}$$

Adding half of (4) to (5) yields

$$3E = 2V_4 + 3V_6 + 4V_8 + \cdots + 3F_3 + 4F_4 + 5F_5 + \cdots \tag{6}$$

Euler's formula for the projective plane is

$$V - E + F = 1. \tag{7}$$

Multiplying (7) by 3 on both sides, substituting the values for V, E and F given by (2), (3) and (6), and simplifying gives

$$V_4 = 3 + V_8 + 2V_{10} + 3V_{12} + \cdots + F_4 + 2F_6 + 3F_6 + \cdots. \tag{8}$$

The values of the F_i's must be non-negative, so

$$V_4 = 3 + V_8 + 2V_{10} + 3V_{12} + \cdots. \tag{9}$$

Dualizing back to the original configuration gives

$$t_2 \geqq 3 + t_4 + 2t_5 + 3t_6 + \cdots. \tag{10}$$

LEMMA. *Given* n *points in the plane,* n *even, then each point must lie on an odd number of lines containing an even number of points.*

PROOF. Let p be a point. There are an odd number of points distinct from p. Each odd line through p contains an even number of points distinct from p, and each even line contains an odd number. Each point distinct from p is on exactly one line through p. Hence an odd number of even lines contains p.

COROLLARY 1. *If* n *is even and exactly two ordinary lines pass through* p, *then* p *lies on at least one even line with at least four points.*

COROLLARY 2. *If* n *is even and a point* p *does not lie on an even line with at least four points, then* p *is on at least one ordinary line.*

A point that lies on exactly two ordinary lines is called a k-*point*. A pair consisting of a point and an ordinary line containing it is called a *flag*. Thus a k-point is on two flags, and since an ordinary line contains two points, the number of ordinary lines is half the number of flags.

In their proof that $t_2 \geqq (3/7)n$, Kelly and Moser establish the inequality

$$t_2 \geqq n/2 - k/6 \tag{11}$$

where k is the number of k-points.

THEOREM. *If* n *is even and* $t_2 \leqq n/2 - 1$, *then*

$$7 \leqq 2t_4 + 3t_6 + 4t_8 + \cdots. \qquad (12)$$

PROOF. If $t_2 \leqq n/2 - 1$ then by (11) $k \geqq 6$. By COROLLARY 1 these k-points are all on even lines with four or more points. There are at least 12 flags on these even lines. The number of points not on these even lines is at least $n - (4t_4 + 6t_6 + 8t_8 + \cdots)$. By COROLLARY 2, each of these points has at least one flag. Therefore the total number of flags is at least

$$12 + n - (4t_4 + 6t_6 + 8t_8 + \cdots).$$

The number of ordinary lines is half the number of flags, so

$$12 + n - (4t_4 + 6t_6 + 8t_8 + \cdots) \leqq 2(n/2 - 1) \text{ or } 14 \leqq 4t_4 + 6t_6 + 8t_8 + \cdots.$$

Dividing by 2 gives the desired result.

2. DETERMINATION OF $t_2(n)$, $n = 14, 16, 18, 22$.

We now derive $t_2(n)$ for $n = 14, 16, 18$, and 22. We need only consider values of t_2 between the lower bound $(3/7)n$ of Kelly and Moser and the upper bound of $n/2$ found by Böröczky

By the above bounds $t_2(14) = 6$ or 7. Suppose $t_2 = 6$. Then (10) becomes

$$6 \geqq 3 + t_4 + 2t_5 + 3t_6 + \cdots. \qquad (13)$$

Another way to write (13) is

$$6 \geqq 2t_4 + 4t_5 + 6t_6 + \cdots. \qquad (14)$$

Comparing (12) and (14) shows that no solution is possible. Since $t_2 = 6$ is impossible, $t_2(14)=7$.

If $n = 16$ then $t_2(16) = 7$ or 8. Suppose $t_2 = 7$ then (1) and (10) become

$$\binom{16}{2} = 120 = 7 + 3t_3 + 6t_4 + 10t_5 + 15t_6 + 21t_7 \qquad (15)$$

$$4 \geqq t_4 + 2t_5 + 3t_6 + 4t_7. \qquad (16)$$

In order for the right hand side of (15) to be divisible by 3, we must have $t_5 = 2$ (since (16) implies $t_5 \leqq 2$). Then (16) implies $t_4 = t_6 = 0$, which contradicts (12). Therefore $t_2(16) = 8$.

The possibilities for $t_2(18)$ are 8 and 9. Suppose $t_2 = 8$. Then we have

$$\binom{18}{2} = 153 = 8 + 3t_3 + 6t_4 + 10t_5 + 15t_6 + 21t_7 + 28t_8 \qquad (17)$$

$$5 \geqq t_4 + 2t_5 + 3t_6 + 4t_7 + 5t_8. \qquad (18)$$

If $t_8 = 1$ then by (18) $t_4 = t_6 = 0$ so (12) cannot be satisfied. Then for the right side of (17) to be divisible by 3, t_5 must be 1. By (17) $t_4 + 3t_6 \leqq 3$ so (12) again cannot be satisfied. Hence $t_2(18) = 9$.

The possibilities for $t_2(22)$ are 10 and 11. Suppose $t_2 = 10$. Then

$$\binom{22}{2} = 231 = 10 + 3t_3 + 6t_4 + 10t_5 + 15t_6 + 21t_7 + 28t_8 + 36t_9 + 45t_{10} \qquad (19)$$

$$7 \geqq t_4 + 2t_5 + 3t_6 + 4t_7 + 5t_8 + 6t_9 + 7t_{10} \qquad (20)$$

If $t_{10} = 1$ then $t_4 = t_6 = t_8 = 0$ and (12) is not satisfied. If $t_8 = 1$ then t_9 must be 1 for the right side of (19) to be divisible by 3. Then $t_4 = t_6 = 0$ and (12) fails. With $t_8 = 0$, t_5 must be 2. Then by (19), $3 \geqq t_4 + 3t_6$ and (12) cannot be satisfied. Therefore $t_2(22) = 11$.

A similar analysis for $t_2(20) = 9$ leaves undecided the case with $t_5 = 1$, $t_4 = 4$, and $t_3 = 49$. This case has the interesting property that (10) becomes an equality. Therefore by (8) all the regions in the dual configuration must be triangular. So far we have not been able to make use of this fact.

Further significant progress on this problem will likely require a new approach. As n increases, the number of undecided cases grows rapidly, and for odd n the formula (12) that is so useful for even n does not apply and there is no corresponding formula that is useful.

REFERENCES

1. D. W. Crowe and T.A. McKee, *Sylvester's problem on collinear points*, Math. Magazine 41(1968), pp. 30-34.
2. L. M. Kelly and W. O. Moser, *On the number of ordinary lines determined by n points*, Canad. J. Math. 10(1958), pp. 210-219.

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