

Instability of the wet cube cone soap film

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Abstract

A “dry” conical soap film on a cubical frame is not stable. Recent experimental evidence seems to indicate that adding liquid to form Plateau borders stabilizes the conical film, perhaps to arbitrarily low liquid fractions. This paper presents numerical simulation evidence that the wet cone is unstable for low enough liquid fraction, with the critical liquid fraction being about 0.000278.

Keywords: soap film, Plateau border, stability, cube, cone.

1 Introduction

A “dry” soap film is the idealization that a soap film has no thickness. Lamarle [1] and Taylor [2] showed that the only singularities possible in dry 2D films in 3D space are three films meeting along a curve at 120° and four triple curves meeting at equal angles at a point, as in the cone over a tetrahedral frame (a *cone* in this paper is the union of a set of line segments from a central point to an outer boundary). Hence the conical film over a cubical frame (figure 1, left) is unstable and jumps to a film with tetrahedral singular points (figure 1, middle).

Real films are “wet” in that they have some thickness and have Plateau borders. A Plateau border is a thickening of the film where several sheets meet, forming concave triangular tubes in place of the triple junctions of dry films. The surface tension on a Plateau border is half that of a dry film, since a dry film is really a double layer. A physical film on a cubical frame that starts wet (figure 1, right) rapidly drains by gravity to a nearly dry film and jumps to the stable dry film, figure 1 middle. But recent laboratory experiments by Hutzler [3], shown in figure 2, seem to show that keeping the Plateau borders from drying out by constantly feeding liquid in from the top stabilizes the wet cone. It is consistent with all these facts that a wet cone is stable for any liquid fraction,

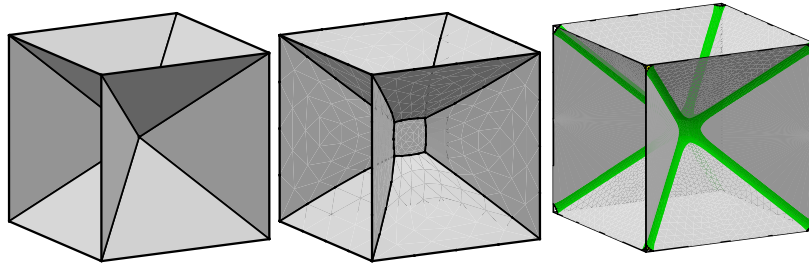


Figure 1: Left: Unstable dry cube cone film. Middle: Stable dry film. Right: Wet cone film with Plateau borders.

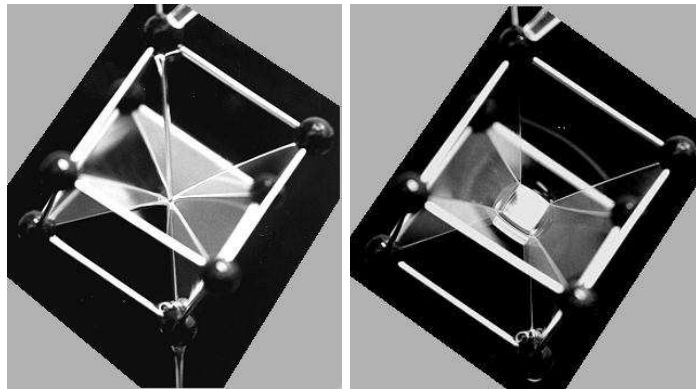


Figure 2: Experiment showing dry cube film (left) and with continuous flow through the Plateau borders (right). (Photos by Stefan Hutzler [3])

but that the barrier to jumping to the stable dry film is low enough that real films rapidly cross it when draining, due to asymmetries or fluctuations. Weaire et al. [4] use a simple model of films and Plateau borders to argue that the wet cone film is stable to arbitrarily low liquid fraction.

The question considered in this paper is whether the wet cube cone film (figure 1, right) is stable to arbitrarily low liquid fractions. The model used is an idealization to the extent that all the liquid is in the Plateau borders, the “dry” part of the film has zero thickness, the wires of the cubical frame have zero diameter, and there are no dynamical effects due to flowing liquid. Also, gravity is neglected.

There are two meaningful alternative conditions to impose on the Plateau borders: constant volume or constant pressure. Constant volume is natural when considering a film on an isolated cubical frame. However, if the film is seen as part of a larger system, a foam for example, then the large volume of liquid in the Plateau borders outside the cube acts as a constant pressure reservoir. It turns out that for the liquid fractions involved in this paper, there

is no significant difference in stability between constant volume and constant pressure.

Drier wet cone films will be more unstable than wetter cones. This is because the Plateau borders are of nearly uniform width for most of their length, so the drier cone essentially contains a shrunken copy of the wetter cone. Hence any unstable perturbation of the wetter cone may be scaled down to an unstable perturbation of the drier cone. Hence there is a critical liquid fraction (possibly 0) above which the wet cone is stable and below which it is unstable. This is not a rigorous argument, but it is compelling enough for the purposes of this paper.

This paper uses the Surface Evolver software (Brakke [5]) to provide two lines of evidence that the wet cube cone is unstable at low liquid volumes. The first line consists of modelling the wet cube cone and finding a deformation that decreases energy. This would seem to be straightforward, but turned out to have considerable difficulties due to the low critical liquid fraction. The second line shows that the central portion of the wet cone in isolation is definitely unstable, and argues that the whole wet cone is still unstable for some low enough liquid fraction. The second line is provided as a backup to the first line, and because it historically preceded the first line by a year and provided the motivation to persevere to find the first line.

Section 2 discusses related results in other dimensions. Section 3 does the deformations of the wet cube cone. Section 4 has the central section argument. Section 5 sketches out the general stability diagram for the wet cube film. Section 6 discusses wetting other cones.

2 Known results in other dimensions

For a 1D film spanning the corners of a square (figure 3), it is known (Bolton et al. [6], Brakke et al. [7]) that if the liquid area is greater than about 0.1 of the area of the square, then the wet cone (figure 3C) is stable. For smaller liquid area, the wet cone is unstable toward jumping to an asymmetric wet cone (figure 3D) or a film with two triangular Plateau borders (figure 3E).

In space dimension 4 and higher, it was shown in Brakke [8], using a technique called *paired calibration*, that the dry conical film over a hypercube frame is stable, in fact absolutely area minimizing. The moral of Brakke [8] is that higher dimension stabilizes cones, since it becomes cheaper for films to meet by going towards the center rather than by going sideways. Thus the wet cube cone should be more stable than the 2D wet cone, possibly even stable down to arbitrarily low liquid fraction.

3 Surface Evolver model of the wet cube cone

The Surface Evolver (Brakke [5]) is a computer program for modelling liquid surfaces shaped by various energies and constraints. A surface has a finite

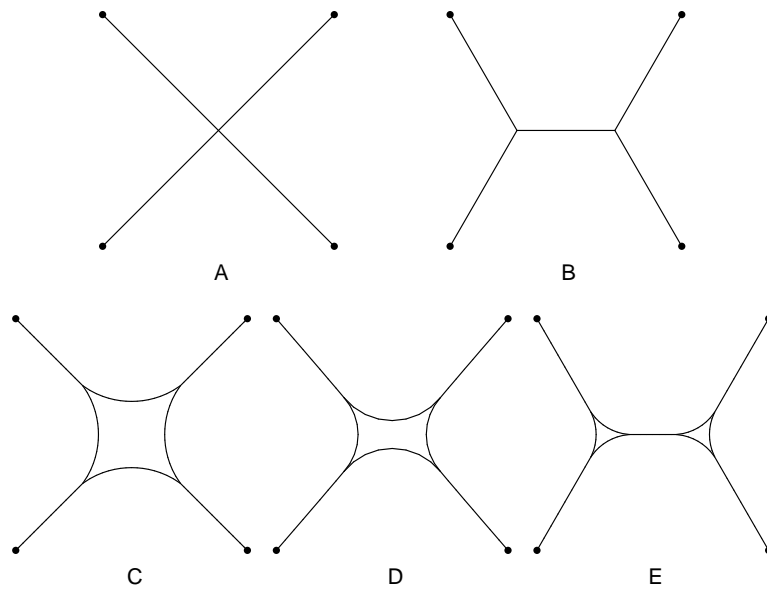


Figure 3: The lower dimensional analog. A. Unstable dry cone in the plane. B. Stable dry film. C. Wet cone at the critical liquid fraction for stability. D. Stable mode for liquid fraction slightly below critical. E. Stable mode for small liquid fractions. B, D and E may also be rotated 90 degrees.

element representation as a set of triangles, with any number of triangles meeting along an edge, so there are no restrictions on surface topology. Either linear or quadratic triangle elements may be used. A surface begins as a crude finite element representation, and is then evolved by refining the triangulation and minimizing energy by gradient descent (augmented with conjugate gradient). The Evolver can also calculate the Hessian matrix of the energy, viewing the energy as a function of all the vertex coordinates. The Hessian can be used to converge quickly to an equilibrium by Newton's Method, and can be used to test the stability of an equilibrium by testing the Hessian's positive definiteness. The Evolver can also calculate the lowest few eigenvalues of the Hessian, which is useful for seeking the critical point of transition between stability and instability. In order to be able to directly compare eigenvalues for different triangulations of a surface, the eigenvalues are defined with respect to an inner product which approximates the L_2 inner product of functions on the surface.

The Evolver works in dimensionless units, so the numerical values reported in this paper can be interpreted in any consistent system of units. Also, since the surface tension is the only energy present in the system, the absolute value of the surface tension is immaterial.

The wet cube cone shares the 48-fold symmetry of the cube, but the presumed perturbation towards a wet version of the stable dry cube film still has 16-fold symmetry. Thus it suffices to make a Surface Evolver model of a fundamental region for the perturbation, shown in figure 4, left. The perturbation involves warping the Plateau border and the dry film on the $x = z$ plane and flattening the volume around the origin while the dry film on the $x = y$ plane remains flat. Figure 4, right, shows a perturbation towards opposite films touching at the center.

The obvious way to test the stability of the 16-fold fundamental region is to let the Evolver calculate the lowest eigenvalue of its energy Hessian. However, calculating Hessian eigenvalues is not perfectly straightforward, as tangential motions of the vertices give many near-zero eigenvalues, even negative eigenvalues in a seemingly well-evolved surface. It is extraordinarily difficult to evolve a highly refined surface to the point of having a positive definite Hessian. Therefore the Evolver has a mode in which only perturbations of the vertices normal to the surface are permitted; these are the perturbations that count in a smooth surface in any case. Unfortunately, the normal-mode Hessian did not show instability, even for liquid fractions down to 0.000008 and highly refined surfaces. As table 1 shows, the lowest eigenvalues look to be heading toward zero with increasing refinement, but never get there, and lower liquid fraction (more negative pressure) hurts instead of helps.

The difficulty turned out to be that a would-be energy-decreasing perturbation really has enough tangential component that forcing it to follow normal vectors causes enough distortion to eliminate the instability. The problem is illustrated in figure 5, which shows a very coarsely refined wet triple junction with free boundaries on a fixed triangular frame. Without normal mode, the film in figure 5A is in neutral equilibrium, since it may be translated within the triangle with exactly equal energy (figure 5B). But with normal mode, trying

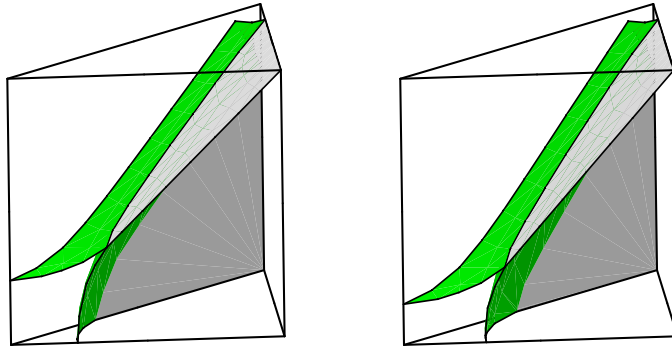


Figure 4: Fundamental region of the wet cube cone for 16-fold symmetry. The front plane of the bounding prism is the $y = 0$ symmetry plane, the back is $x = y$, the bottom is $z = 0$, the top is $z = 1$, and the right is $x = 1$. The outer end of the Plateau border is fixed on an ideal cross-section perpendicular to the border axis, to provide a well-behaved termination of the border. The liquid fraction shown is 0.023. Left, the symmetric equilibrium. Right, the perturbation towards the flat central film.

to translate using only normal motions distorts the Plateau border, raising the energy (figure 5C). A little dimensional analysis shows that eigenvalue grows inversely with the diameter of the Plateau border, so the problem gets worse and worse with smaller liquid fraction.

The cure turned out to be to define a direction of motion in the direction the perturbation wanted to go, rather than the normal direction. I added a special Hessian mode to the Evolver that allows motion only in a prescribed direction at each vertex. Using a direction field of the form $(x, y, -\alpha z)$ suppressed all the unwanted tangential motion, but allowed the desired perturbation. This mode finally revealed negative eigenvalues. Some experimentation showed that

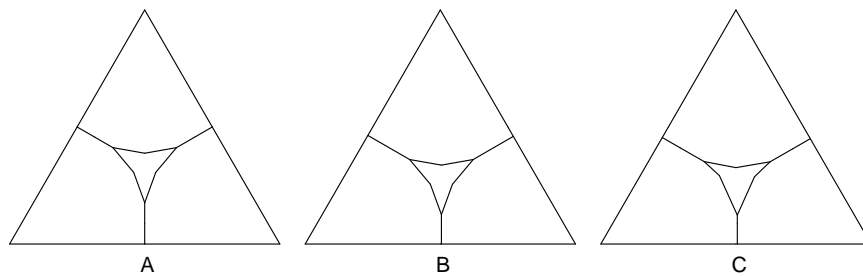


Figure 5: Distortion of Plateau border by normal motion. A: Undistorted version. B: Undistorted after motion. C: Distorted after normal motion; note the uneven breaks in two of the Plateau border sides.

Pressure	Linear 1	Linear 2	Linear 3	Quad 1	Quad 2	Quad 3
-5	2.2796	2.0822	1.9371	2.1804	1.9265	1.8026
-10	3.1058	2.5252	2.0527	2.5305	1.8821	1.5736
-20	5.3673	3.9835	2.8712	3.8642	2.2978	1.5148
-30	7.1128	5.3430	3.8983	5.1192	2.8688	1.6063
-40	7.9647	6.6114	4.7987	6.4680	3.8075	2.0919
-50	9.2277	7.3067	5.5445	7.1024	4.1854	2.0186
-60	9.7954	8.2182	6.2913	7.8381	4.7170	2.1838
-70	10.234	8.9036	6.8698	8.5718	5.4999	2.6150

Table 1: Lowest eigenvalues for various Plateau border pressures and various levels of refinement for linear and quadratic facets, using the standard normal motion restriction for the Hessian. On one hand, the eigenvalues go up with more negative pressure, but they are dropping drastically with increasing refinement. So the conclusion is not at all clear.

Pressure	Linear 1	Linear 2	Linear 3	Quad 1	Quad 2	Quad 3
-5	1.1395	1.0612	1.0338	1.0424	1.0146	1.0080
-10	0.9921	0.8785	0.8468	0.8432	0.8211	0.8155
-20	0.7860	0.5798	0.5233	0.5403	0.4740	0.4634
-30	0.4211	0.2851	0.2348	0.2680	0.1193	0.0955
-40	0.1529	-0.0451	-0.1066	-0.1309	-0.2729	-0.2977
-50	-0.2786	-0.4418	-0.4257	-0.3410	-0.6835	-0.7311
-60	-0.4986	-0.7026	-0.6828	-0.8248	-1.1737	-1.2297
-70	-0.8693	-1.1772	-1.1445	-1.3767	-1.7027	-1.7637

Table 2: Lowest eigenvalues for various Plateau border pressures and various levels of refinement for linear and quadratic facets for special Hessian direction field (x,y,-4z). Eigenvalues for constant volume are practically identical.

$\alpha \approx 4$ gave the lowest eigenvalues. Table 2 shows the lowest eigenvalues for various fixed pressures. Trials near the critical liquid fraction showed the critical pressure to be about -32.4, with critical liquid fraction about 0.000278.

4 Central region with free boundary

Since modelling a full wet cube cone (or even a fundamental region of it) is so difficult, I originally decided to try to isolate the inner part of the wet cone from the outer part. The inner part should be pretty much the same for all liquid fractions if scaled correctly, and the outer part should be just small perturbations of flat planes and Plateau borders with cylindrical surfaces. To that end, I truncated the inner portion with an octahedron (figure 6), with the film boundary free to move on the octahedron. I reasoned this would be the least

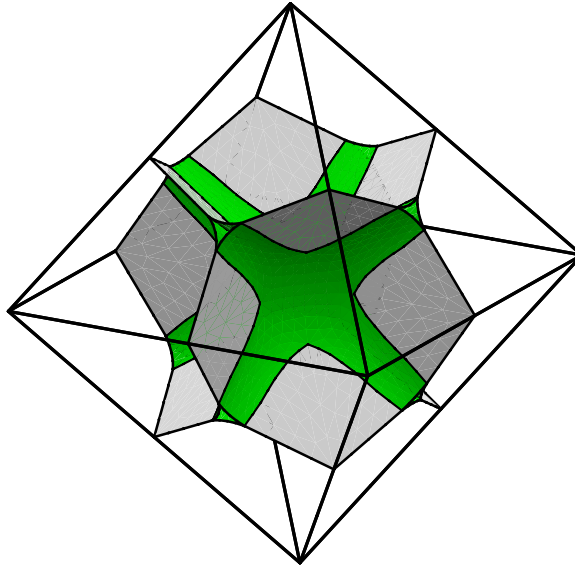


Figure 6: Central region with free boundaries on an octahedron, in symmetric equilibrium.

disruptive way to isolate the central region, in that for the dry film the central point can translate arbitrarily within the octahedron without area change, since the films remain orthogonal to the octahedron. The constraint on the Plateau borders is constant pressure, since even if the whole wet cone is at constant volume, the outer Plateau borders (being much larger than the central region) serve as a constant pressure reservoir.

Surface Evolver calculations show that this central region by itself is unstable, even without the special Hessian direction field. In fact, the eigenspace for the one negative eigenvalue has dimension 2. This might seem strange, since there are three ways for the cone to change to the stable dry film, but they turn out to be linearly dependent. The infinitesimal generators of the three ways are essentially like

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (1)$$

To get more precise control over the boundary on the octahedron, in order to match it to perturbations of the outer regions, I made a version of the central region with a fixed boundary trace on the octahedron. This trace consists of straight segments and circular arcs of radius $1/p$ around the Plateau border, where p is the difference of external pressure over internal pressure. This trace has an adjustable parameter in it that moves it in the direction of the unstable perturbation. This enables making a known perturbation of the form $(h, h, -2h)$

Pressure	Facets	Coefficient	Critical volume
2	21	-0.8211	0.000055
	84	-0.8605	
	336	-0.8821	
	1219	-0.8905	
4	37	-0.9504	0.000065
	148	-1.0729	
	592	-1.1100	
	2323	-1.1383	
8	57	-1.1653	0.000066
	228	-1.3477	
	819	-1.4338	
	3268	-1.5205	

Table 3: Coefficients of quadratic term in energy as function of perturbation of fixed boundary. Higher pressures correspond to a larger octahedron relative to the cube, hence thinner Plateau borders in the octahedron. Critical volume values are from equation 12; these are lower bound estimates on the critical liquid fraction for the cube.

uniformly over the entire surface, re-minimize the surface using the Hessian and Newton's Method, and thus numerically measure the coefficient of the quadratic term in the energy as a function of h . The value of h actually used was 0.00001, small enough not to distort the triangulation. The results are presented in table 3.

So the question is now: Is the area loss of the unstable deformations of the inner region compensated for by the area increase of the deformation of the outer regions connecting the inner region to the cube frame? I present the following argument (not a proof yet) that it isn't, for low enough volume.

The deformation of the inner region bends the large flat outer films away from flatness. For small deformations, this outer perturbation is nearly a harmonic function, since a nearly flat minimal surface is nearly the graph of a harmonic function. The dominant term in this harmonic function ought to be $\log(r)$, since the deformation is largest toward the center of the cube. If the cube frame is at distance D from the center, then the area excess of the deformation turns out to be proportional to $1/\log(D)$. Hence for large enough D , the central area decrease will win.

More precisely, consider the following specific deformation to connect the perturbed boundary to the outer frame. Let the inner octahedron face be $x + y + z = a$ and let the cube corner be at (D, D, D) . Then the octahedron face $x + y + z = 2D$ is still within the cube, at least the part containing the film. Consider deformations that on the octahedron face $x + y + z = t$ slide the section by

$$h(t)(1, 1, -2), \quad h(2D) = 0, \quad (2)$$

with the accompanying stretching and shrinking of the dry film on the face edges, as in figure 5B. The perturbation is taken to be zero between $x + y + z = 2D$ and the cube frame. Note this perturbation preserves Plateau border volume exactly, and on each slice the net change in cross-section length is zero. So the only area increase comes from the bending of the surface. The bending of the Plateau border contributes less excess than would the bending of the dry film (easy calculation), so we just compute the area excess of the dry film.

To flatten out the domain, make the change of variables to an orthonormal system of coordinate (u, v, w) defined by

$$u = (x + y + z)/\sqrt{3}, \quad v = (x - 2y + z)/\sqrt{6}, \quad w = (z - x)/\sqrt{2}. \quad (3)$$

The dry film subject to our perturbation is then defined by

$$a/\sqrt{3} \leq u \leq 2D/\sqrt{3}, \quad 0 \leq v \leq u/\sqrt{2}, \quad w = 0. \quad (4)$$

The magnitude of the perturbation is

$$w(u, v) = \frac{3}{\sqrt{2}} h(u). \quad (5)$$

The area is (since film is double density here)

$$A = 2 \int_{a/\sqrt{3}}^{2D/\sqrt{3}} \int_0^{u/\sqrt{2}} \sqrt{1 + w_u^2} dv du, \quad (6)$$

so the excess area is

$$\begin{aligned} Excess &= 2 \int_{a/\sqrt{3}}^{2D/\sqrt{3}} \int_0^{u/\sqrt{2}} \sqrt{1 + w_u^2} - 1 dv du \\ &\approx 2 \int_{a/\sqrt{3}}^{2D/\sqrt{3}} \int_0^{u/\sqrt{2}} \frac{1}{2} w_u^2 dv du \\ &= \int_{a/\sqrt{3}}^{2D/\sqrt{3}} \frac{u}{\sqrt{2}} \frac{9}{2} h'(u)^2 du. \end{aligned} \quad (7)$$

The calculus of variations shows that the optimal form of $h(u)$ here is

$$h(u) = h_0 \frac{\log(2D/\sqrt{3}) - \log u}{\log(2D/\sqrt{3}) - \log(a/\sqrt{3})}, \quad (8)$$

where $h_0 = h(a/\sqrt{3})$ is the perturbation at the inner octahedron. Plugging this back in gives

$$\begin{aligned} Excess &= \frac{9}{2\sqrt{2}} \int_{a/\sqrt{3}}^{2D/\sqrt{3}} u h_0^2 \frac{1}{u^2} (\log(2D/\sqrt{3}) - \log(a/\sqrt{3}))^{-2} du \\ &= \frac{9}{2\sqrt{2}} \frac{1}{\log(2D/a)} h_0^2. \end{aligned} \quad (9)$$

So if D is large enough, the excess in outer area is less than the area loss in the inner region. In particular, if we let C be the coefficient from table 3, with $a = 2$, then the critical value of D occurs when

$$C + \frac{9}{2\sqrt{2}} \frac{1}{\log D} = 0, \quad (10)$$

or

$$D = \exp(-9/2\sqrt{2}C). \quad (11)$$

To get the corresponding liquid fraction, multiply the length of the Plateau border, $\sqrt{3}D$, by its cross section for a given radius r , $(\sqrt{3} - \pi/2)r^2$, and divide by the cube octant volume:

$$V_{crit} = \frac{\sqrt{3}D(\sqrt{3} - \pi/2)r^2}{D^3} = \sqrt{3}(\sqrt{3} - \pi/2)r^2 \exp(9/\sqrt{2}C). \quad (12)$$

These numbers are in the last column of table 3. Given the restricted nature of the perturbations considered, these values only give a lower bound on the true critical volume. But they are only a factor of four lower than the best estimate of $V_{crit} = 0.000278$, so this approach does seem to provide a good picture of what is happening and why the critical volume fraction is so low. Whenever logs and exponentials appear, numbers can get extreme very easily.

5 Wet cube film stability diagram

A stability diagram is a handy way to describe the behavior of a system. For simplicity, we first consider the system consisting of 1/16 of a wet cube film. The control parameter will be the liquid liquid fraction, and the response will be the asymmetry of the film, measured as the difference in the volumes of the two exterior regions of the film. This stability diagram will be rather qualitative, so exact definitions are not needed. The stability diagram is sketched in figure 7.

Solid lines represent stable films, and dashed or dotted lines represent unstable films. Point D is the unstable dry cube cone. Point C is the critical volume symmetric wet cube cone. The dashed straight line DC represents the unstable symmetric wet cube cones. The solid line CE represents the stable wet cones. Point A is the stable dry cube film. The curve AB represents progressive wetting of the the dry cube film, until it reaches the critical point B, where the central film disappears. The dotted line DBG is the transition to a central film, where opposite sides of the central region first touch. These are not equilibria, except at point B. Evolver simulations put the liquid fraction for point B at 0.0103. The dashed curve CBF represents unstable equilibrium asymmetric wet films, combinatorially the same as the wet cube cone. On segment BF, opposite films pass through each other in the central region, so this segment is not physical. The lower dashed curve DC represents equilibria reached by going downhill in energy from the symmetric unstable wet cone in the opposite direction from the central film. These films have the central region stretched

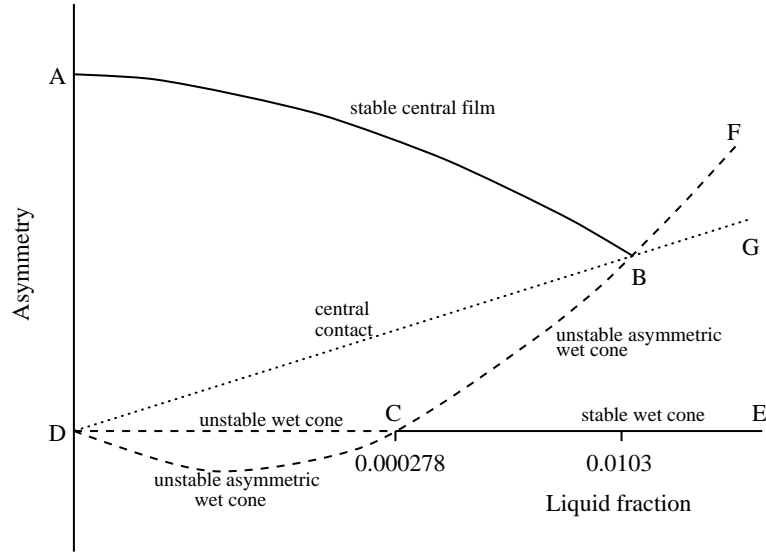


Figure 7: Qualitative stability diagram of wet cube film for fixed volume. Not to any kind of scale whatsoever.

vertically into a pillar rather than flattened into a slab. They are stable when 16-fold symmetry is imposed, but otherwise are unstable to jumping to vertical central films. This curve terminates at point D, the dry cone, because there is no equilibrium dry film with a fourfold junction curve as pictured in figure 8. This nonexistence follows from a calibration argument that the dry cube cone is the unique absolute minimum in a certain wedge, and that any equilibrium film in the wedge that is the graph of a function can be calibrated, and is hence an absolute minimum.

The details of the diagram around point B deserve a more detailed discussion. The central film configuration can be viewed as a perturbed wet cone, as in figure 4, right, with the perturbation big enough for the central films to overlap, and then the overlaps pushed back to the midplane. If an overlap is pushed back distance z , the work done may be approximated as follows: At internal pressure $-P$, the overlap is approximately spherical of radius $R = 2/P$. The plane of contact at pushback z will be approximately the area of a spherical slice, which is $A = 2\pi Rz$. The pushing force is $F = PA = 4\pi z$. The total work done pushing is $W = 2\pi z^2$. Since there are top and bottom films to push back, the total work done is really $W = 4\pi z^2$. Now suppose the wet cone has its unstable equilibrium at $z = z_0$, and its energy is $E(z) = E_0 - c(z - z_0)^2$. Because nothing unusual happens at point B if one lets the films interpenetrate, z_0 is a smooth function of liquid fraction. Evolver computations show that $c \approx 0.075$. The total energy of the central film configuration formed by taking some wet cone

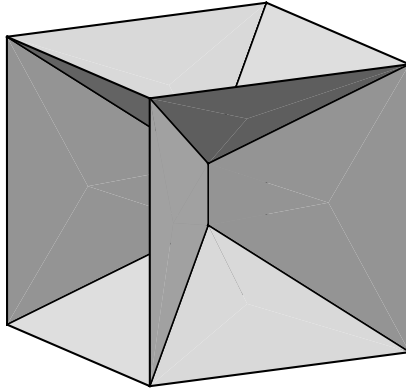


Figure 8: Dry film version of the wet films on the “other side” of the wet cone saddle point from the central square, i.e. the lower curve DC of figure 6. No dry film of this shape actually exists, however.

and pushing back will be

$$E_{net}(z) = E_0 - c(z - z_0)^2 + 4\pi z^2. \quad (13)$$

This has its minimum at

$$z = \frac{-2cz_0}{8\pi - 2c}. \quad (14)$$

This shows that curve AB does in fact meet DG where CF crosses, because $z = 0$ where $z_0 = 0$. Since z_0 is a nearly linear function of liquid fraction near point B, the central film line AB is nearly linear at B, and it does not go vertical. Furthermore, the eigenvalue for the central film does not approach zero at B. This seems to be confirmed by the few Evolver tests I’ve done on it, although given all the perils of eigenvalue computing mentioned in this paper, I make the claim cautiously.

The diagram for constant pressure is very similar. The equilibria are the same, and the only difference is that the lines of constant pressure deviate slightly from the lines of constant volume (which are vertical in figure 7). For the same Plateau volume, the central film configuration has a slightly less negative internal pressure than the wet cone. At the dry extreme, this follows from the fact that the total length of the Plateau border in the dry cone is $8\sqrt{3} \approx 13.856$, while the length in the stable dry film is about 13.746, according to the Evolver. At liquid fraction 0.098 (near point B), the Evolver gives the pressure in the wet cone as -5.976, but with the central film -5.64.

The stability diagram for the full wet cube cone can best be visualized by taking the response space to be the two-dimensional eigenspace for the lowest eigenvalues found in section 4. The diagram contains three copies of figure 7 spread at 120° around a common axis *DCE*.

6 Other wet cones

By the results of Lamarle [1], Heppes [9], and Taylor [2], there are eight possible dry cone singularities (not counting the triple junction of planes). The cone over the tetrahedron is stable when dry, and therefore when wet. We have seen the wet cube cone can be unstable. Of the other six cones, it turns out from Evolver experiments that only the cone over the dodecahedron even has an equilibrium wet cone. For the five others, trying to form a wet cone results in a surface that has an energy gradient in the direction of the wet version of the minimal dry film. This does not happen with the cube and dodecahedron because their high symmetry rules out nonzero gradients.

By Evolver experiments of the same type as in section 4, the wet dodecahedral cone inner region turns out to be unstable, with negative eigenvalue eigenspaces of dimension 2 and 3, for a total of 5 unstable degrees of freedom. No doubt the full dodecahedral wet cone is unstable below some critical liquid fraction.

7 References

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